Constrained discrete-time nonlinear controller for a fluid tank system

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Abstract
This paper basically follows [1], where the transfer function formalism in nonlinear control systems, both continuous and discrete-time, is discussed. Here, we demonstrate how such a formalism can be adopted to design constrained nonlinear discrete-time controller for a fluid tank system. The first step consists in finding appropriate nonlinear discrete-time description for a fluid tank system, which is, in general, a difficult task, as it involves solving nonlinear differential equations. Second, we design nonlinear discrete-time controllers which linearize closed loop, eliminate input disturbance and deal with the controller output constraint. Finally, all results are verified on a real fluid tank system.

Keywords: nonlinear systems, transfer functions, skew polynomials, constrained controller design, fluid tank system

Introduction

Algebraic formalism in nonlinear control systems, both continuous [2] and discrete-time [3], shows great applicability in solving a number of control problems, like decompositions to canonical forms, feedback linearization, disturbance decoupling problem, to name a few possibilities. A power of such a formalism was recently extended by introducing transfer functions of nonlinear systems, see [4], [5] and [6] for special cases of continuous and, respectively, discrete-time systems. An overview of such a formalism was given in [1]. So we refer the reader to that paper for detailed mathematical background which we, to avoid duplicity, will leave out in this work.

Transfer functions of nonlinear control systems, as defined in [4] and [6], show many properties we expect from transfer functions. One of them is the possibility to use transfer function algebra when combining systems in a series, a parallel or a feedback connection. In this paper, this feature is employed to design various nonlinear discrete-time controllers for a fluid tank system. In controller design we will be interested in controllers which satisfy linearity of the closed loop, eliminate an input disturbance and also deal with the controller output constraint.

1. Algebraic background

Algebraic formalism for dealing with nonlinear discrete-time control systems was given in [4] and recently extended in [6] by introducing transfer functions of nonlinear discrete-time systems. Some former results concerning transfer functions are also due to [7]. Here, we just briefly review some necessary facts and the reader is referred to [4], [6] for detailed technical constructions which are not found here.

For the sake of simplicity we introduce following notation. For any variable $\xi(t)$ we write only ξ and for its time shift $\xi(t+1)$ we use ξ^+.

Using this notation, nonlinear discrete-time control systems considered in this paper are objects of the form

$$\begin{aligned} x^+ &= f(x,u) \\ y &= g(x,u) \end{aligned} \tag{1}$$

where *f* and *g* are meromorphic functions, which we think of as elements of the quotient field κ of the ring of analytic functions and *x*, *u* and *y* denote state, input and output to the system and are of appropriate dimensions.

We suppose that (1) is generically submersive [3].

Define a difference vector space ε of one-forms spanned over κ by differentials of elements of κ

$$\varepsilon = \operatorname{span}_\kappa\{d\xi; \xi \in \kappa\}$$

Let σ be a time shift operator which takes *t* to *t* + 1. Operator σ acts on κ and ε as follows

$$\sigma(\xi) = \xi^+$$

$$\sigma(\alpha d\xi) = \alpha^+ d\xi^+$$

for any $\xi \in \kappa$ and $\alpha d\xi \in \varepsilon$.

Operator σ induces a skew polynomial ring $\kappa[z;\sigma,0]$ in the indeterminates *z* with the usual addition and (non-commutative) multiplication given by the commutation rule

$$z\varphi = \varphi^+ z$$

for any $\varphi \in \kappa$. Hence, $\kappa[z;\sigma,0]$ represents the ring of linear shift operators which act over the vector space ε [6]. That is, the shift operator σ on ε induces the action

$$*: \kappa[z;\sigma,0] \times \varepsilon \to \varepsilon; \left(\sum_{i=1}^{n} a_i z^i\right) * u = \sum_{i=0}^{n} a_i \sigma^i(\upsilon)$$

for any υ in ε. For the sake of simplicity, the symbol * is often dropped.

Moreover, $\kappa[z;\sigma,0]$ is a left Ore ring and can be therefore embedded to the non-commutative quotient field $\kappa\langle z;\sigma,0\rangle$. After defining quotients of skew polynomials [6] transfer functions can be introduced.

Given the nonlinear system (1) with *m* = 1 and *p* = 1. An element $F(z)\in\kappa\langle z;\sigma,0\rangle$ such that $\mathrm{d}y = F(z)\mathrm{d}u$ is said to be a transfer function of discrete-time nonlinear system (1).

Transfer functions of nonlinear discrete-time systems have many properties we expect from transfer functions. For instance, one can easily introduce algebra of transfer functions when combining systems in a series, parallel or feedback connection. Such a formalism will be used later in controller design.

2. Nonlinear discrete-time model of the fluid tank system

Properties of continuous-time systems are usually described by differential equations while for discrete-time systems we use difference equations. In case the systems are linear we can to advantage use *Z* transformation to find discrete-time models from continuous. However, if we deal with nonlinear systems, the situation is much more difficult to handle. Since to find nonlinear discrete-time description of continuous-time system one needs to solve nonlinear differential equations and then sample the solution by a sampling period *T*. Of course, this is usually impossible due to the nonlinear relations. Hence, in finding nonlinear discrete-time descriptions, one has to be satisfied only with approximations. This idea is employed to describe properties of a fluid tank system.

Consider the fluid tank system described by the following continuous-time state-space representation

$$\begin{aligned}\dot{x} &= \frac{1}{A}u - c\sqrt{x}\\ y &= x\end{aligned} \tag{2}$$

where *x*, *A* and *c* denote a level of a liquid, a tank area and a flow coefficient, respectively.

Due to the nonlinear relations, we are not able to find any solution to (2). Hence, to find a nonlinear discrete-time model of (2) we use the following approximation

$$\dot{x} = \frac{\mathrm{d}x(t)}{\mathrm{d}t} \approx \frac{\Delta x(t)}{\Delta(t)}$$

Clearly, we can think of Δt as a sampling period *T* which implies that $\Delta x(t) = x(t+1) - x(t)$. That is

$$\dot{x} \approx \frac{x^{+} - x}{T}$$

Hence, from (2) we get the following nonlinear discrete-time approximation

$$\begin{aligned}x^{+} &= x + \frac{T}{A}u - cT\sqrt{x}\\ y &= x\end{aligned} \tag{3}$$

Obviously, the less sampling period *T* we choose, the more accurate approximation we get.

Now, the transfer function of (3) can be computed as

$$\mathrm{d}x^{+} = \mathrm{d}x + \frac{T}{A}\mathrm{d}u - \frac{cT}{2\sqrt{x}}\mathrm{d}x$$

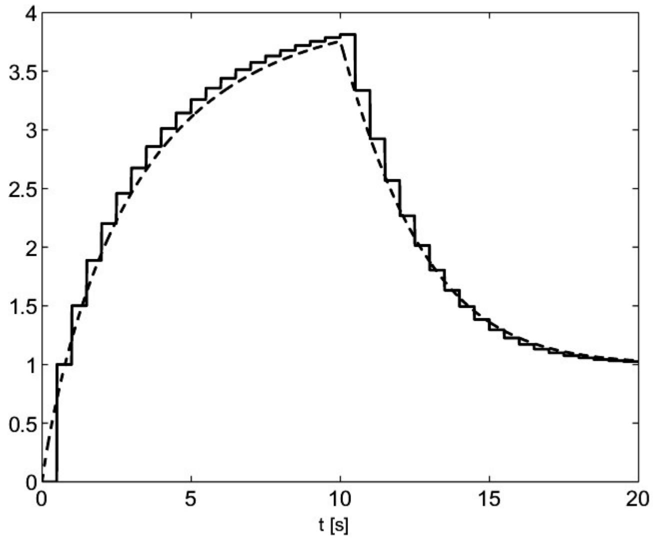

Fig.1 Transient responses: nonlinear discrete-time approximation of the fluid tank system with *A*=1, *c*=1 and *T*=0.5*s*

$$(z - 1 + \frac{cT}{2\sqrt{x}})\mathrm{d}x = \frac{T}{A}\mathrm{d}u$$

and

$$F(z) = \frac{K}{z - D} \tag{4}$$

where $K = \frac{T}{A}$ and $D = 1 - \frac{cT}{2\sqrt{x}}$

3. Constrained discrete-time nonlinear controller

In this section we use the discrete-time representation (3) and (4) of the fluid tank system and design nonlinear discrete-time controller. Our aim is to design a controller, which:

- satisfies a linear transfer function of the closed loop
- eliminates an input disturbance
- deals with a control signal constraint

The requirement of the closed loop linearity can be satisfied by a feedback controller. This yields a feedback linearization. If we want the closed loop dynamics to be determined by a time constant T_1 we obtain regular static state feedback

$$u = [(1 - D_1)w + cT\sqrt{x} - (1 - D_1)x]\frac{A}{T} \tag{5}$$

where *w* denotes the new input and $D_1 = e^{-T/T_1}$. Under this feedback the input-output description of the closed loop is linear

$$\mathrm{d}y = \frac{1 - D_1}{z - D_1}\mathrm{d}w$$

Evidently, to linearize the system, we did not need the introduced transfer function formalisms. However, the situation is different in eliminating an input disturbance and dealing with a control signal constraint, where a use of transfer functions is unavoidable.

To satisfy remaining design requirements we will consider control structure [8] depicted in Fig. 2.

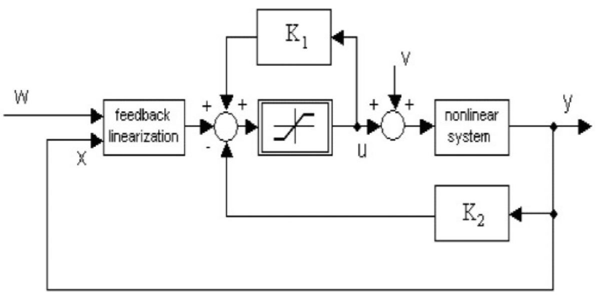

Fig.2 Closed loop structure

The input disturbance *v* is eliminated via the feedback filter K_2 which tries to reconstruct *v* and subtracts it from controller output. The block K_1 only removes the impact of K_2 while controlling the system (via the feedback linearization). It is worth noting that the entire control structure has the properties of a PI controller. An important difference consists in the fact that there is no problem with the wind-up effect, in contrast to the classical PI controller with the control signal constraint.

Of course, the ideal filter $K_2(z) = \frac{1}{F(z)}$ is not realizable. Hence, we use

$$K_2(z) = \frac{(1-\lambda_f)(z-D)}{(z-\lambda_f)K} \tag{6}$$

where $\lambda_f = e^{-T/T_f}$ and T_f is a time constant which determines how fast the disturbance elimination will be.

The transfer function (6) corresponds to the input-output difference equation

$$K(z-\lambda_f)\mathrm{d}y_{K2} = (1-\lambda_f)(z-D)\mathrm{d}u_{K2}$$

$$K(y_{K2}^{+} - \lambda_f y_{K2}) = (1-\lambda_f)(u_{K2}^{+} + cT\sqrt{u_{K2}} - u_{K2})$$

where u_{K2} and y_{K2} denote input and output respectively to the filter.

The realization (state-space description) can be found as

$$x_{K2}^{+} = \lambda_f x_{K2} + \frac{1-\lambda_f}{K}(\lambda_f u_{K2} + cT\sqrt{u_{K2}} - u_{K2})$$

$$y_{K2} = x_{K2} + \frac{1-\lambda_f}{K} u_{K2}$$

The filter $K_1(z)$ is linear system with the transfer function

$$K_1(z) = \frac{1-\lambda_f}{z-\lambda_f} \tag{7}$$

3.1 Simulation results

It is quite important to note that all controllers were designed using the nonlinear discrete-time approximation (3) of the fluid tank system (2). But as a matter of fact the continuous-time system (2) is to be controlled. And since (3) is only its approximation, more or less accurate, the quality of the closed loop transient responses highly depends on the chosen sampling period *T*. This is something which constitutes a fundamental difference with respect to the linear case and it is due to the fact that we were not able to find the discrete-time model of the system (2), only its approximation.

In simulations we used the system (2) with *A*=1, *c*=1. Time constants T_1 and T_f which determine the dynamics of the control and of the disturbance elimination were chosen to be 0.5*s* and 0.2*s* respectively. The control signal was constraint to the interval $\langle 0;2 \rangle$. Transient responses for different sampling periods *T* are depicted in Fig. 3 and Fig.4.

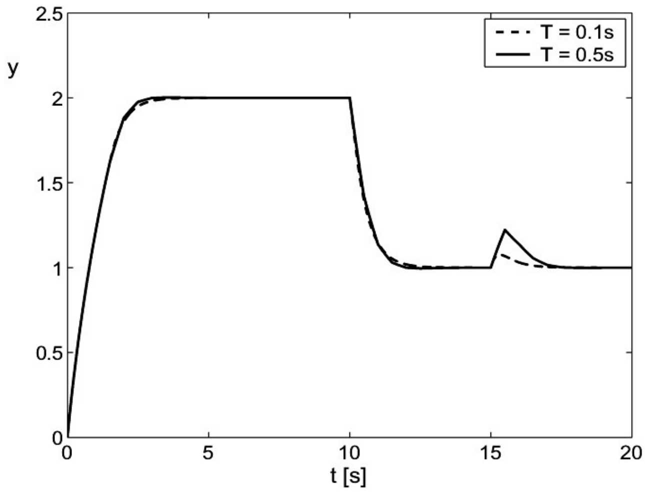

Fig.3 System output. Disturbance step from 0 to 0.5 in time 15*s*

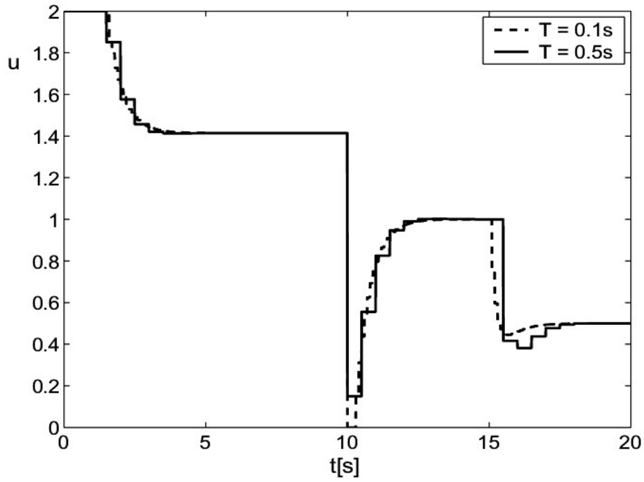

Fig.4 Controller output. Disturbance step from 0 to 0.5 in time 5*s*

4. Minimum-time controller

Minimum-time controllers can be obtained quite easily from (5), (6) and (7) by requiring $T_1 \to 0$ and $T_f \to 0$ which implies that $D_1 \to 0$ and $\lambda_f \to 0$. As a result we get the following regular static state feedback

$$u = (w + cT\sqrt{x} - x)\frac{A}{T}$$

and filters

$$K_1 = \frac{1}{z}$$

$$K_2 = \frac{z-D}{Kz}$$

The latter corresponds to the state-space realization

$$x_{K2}^{+} = \frac{1}{K}(cT\sqrt{u_{K2}} - u_{K2})$$

$$y_{K2} = x_{K2} + \frac{1}{K}u_{K2}$$

Such controllers represent in fact dead-beat controllers. We can easily check that

$$\mathrm{d}y = \frac{1}{z}\mathrm{d}w$$

$$\mathrm{d}y = \frac{1}{z}\mathrm{d}v$$

Of course, as we have a constrained control signal, transient responses will not take only one sampling interval, but spend a necessary time on the constraints. Simulations are depicted in Fig. 5 and Fig. 6. We use the same parameters but T_1 and T_f, which are now zero.

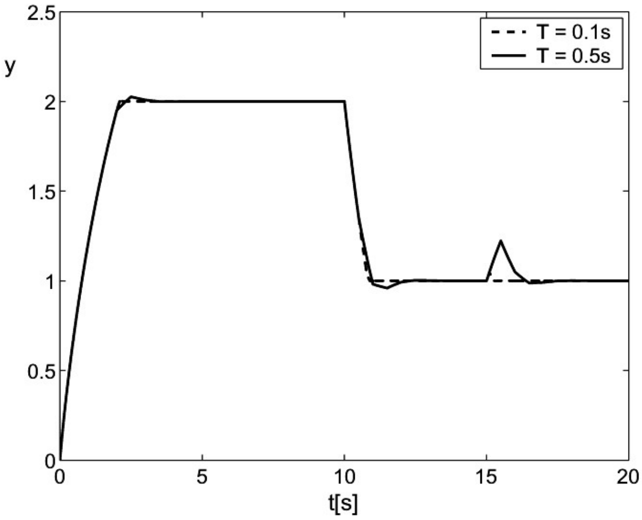

Fig.5 System output. Disturbance step from 0 to 0.5 in time 15*s*

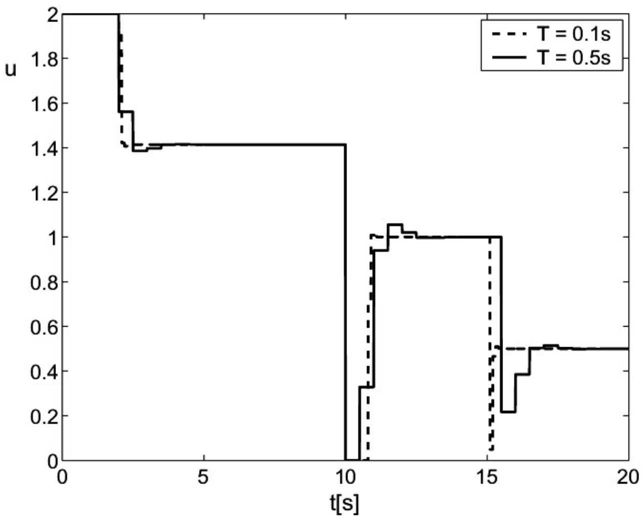

Fig.6 Controller output. Disturbance step from 0 to 0.5 in time 15*s*

Another important thing to note is that after leaving the constraints the control signal does not immediately achieve the steady value, as it would in the case of a linear system. This is caused again by using the discrete-time approximation (3) of the system (2), as we were not able to find the discrete-time model of the system (2).

For the same reason, also here the quality of the closed loop transient responses highly depends on the chosen sampling period *T*.

5. Real fluid tank system

The construction of the real hydraulic system which is depicted in Fig. 7 is due to [9]. It enables us to control the three-tank cascade.

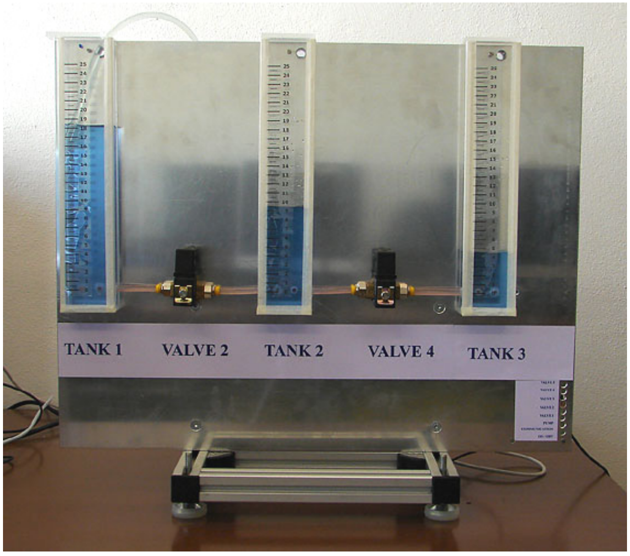

Fig.7 Three-tank cascade

The first step consists in the identification. We are interested in controlling the first tank. Identified parameters are as follows: $A = 0.001\mathrm{m}^2$, $c = 0.0141$. Fig. 8 depicts differences between the identified model and the real fluid tank system.

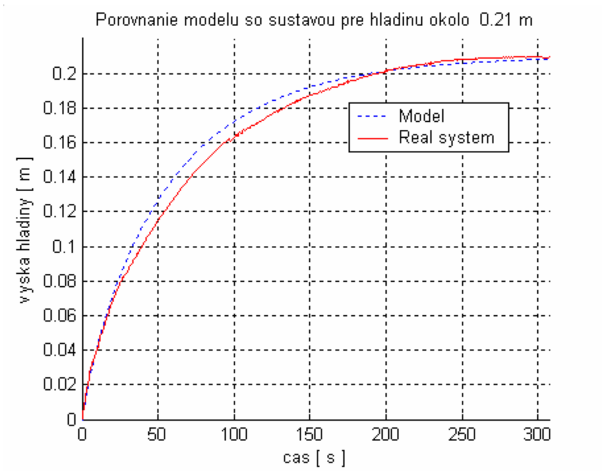

Fig.8 Differences between model and the real system

Note again that controllers were designed using the nonlinear discrete-time approximation of the fluid tank system. But in fact the continuous-time system is to be controlled. Hence, the less sampling period *T* we choose, the better results we get. Obviously, this has a significant impact on the quality of the control of the real fluid tank system.

Another important thing to note which is directly related to the control of the real system is the fact that when dealing with the minimum-time control the results will differ from the simulations, especially in the case of the controller output. This is caused by the controller which when detecting the difference between system output and its required value presses for its eliminating within one sampling interval. Clearly, this results in the maximum value of the controller output which is, however, constrained. Such a situation happens almost permanently in all real processes, at least owing to the noise of measured signals, sensors, etc. Hence, the minimum-time controller will switch the pump on and off very frequently, depending on the chosen sampling period *T*. An additional filtering or a dead zone can be solutions to the problem which is, however, out of scope of this paper.

In the real fluid tank simulations we used the step of the required value from 0 *m* to 0.2 *m* at the beginning and from 0.2 *m* to 0.1 *m* in time 100 *s*. The disturbance appeared in time 150 *s*.

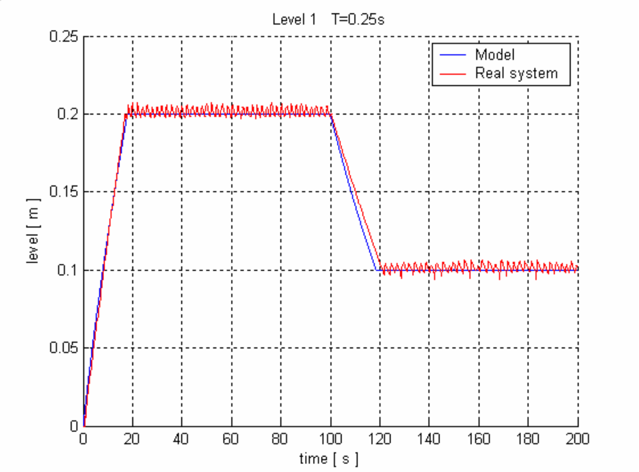

Fig.9 System output in minimum-time control with *T*=0.25*s*

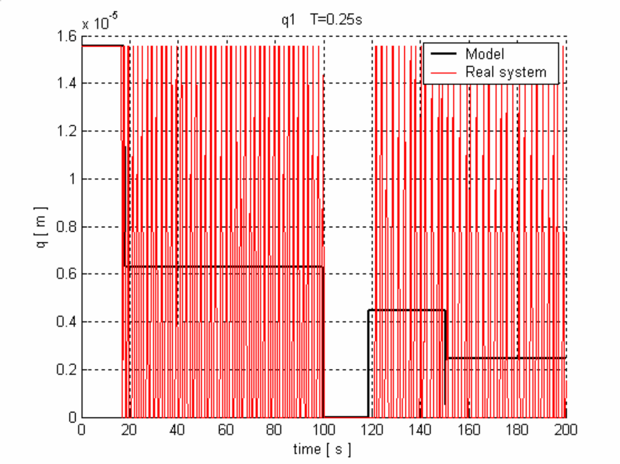

Fig.10 Controller output in minimum-time control with *T*=0.25*s*

In the non-minimum-time control we used time constants $T_1 = 10$ *s* and $T_f = 4$ *s* which characterize how fast the control and, respectively, the disturbance elimination will be. Here, we can see quite high quality of the control, the disturbance elimination and the controller output as well. The difference between controller output of the model and of the real system is due to the nonlinear properties of the valve as well as to the viscosity of the pump.

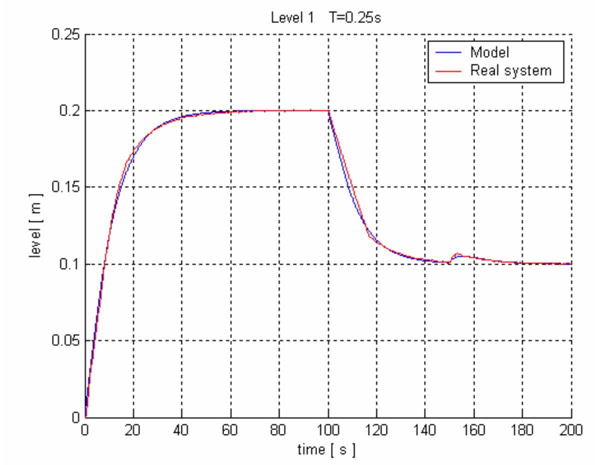

Fig.11 System output in non-minimum-time control with *T*=0.25*s*

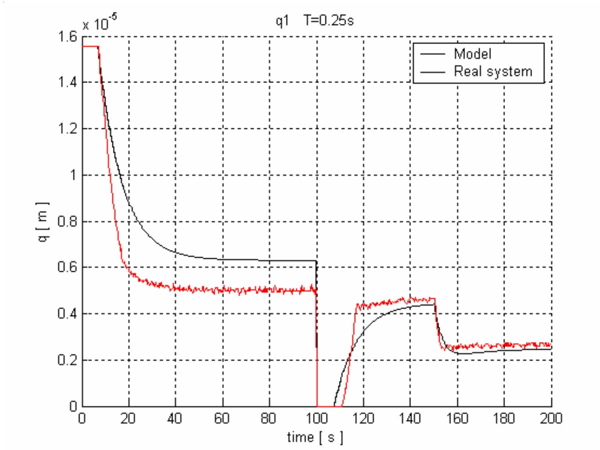

Fig.12 Controller output in non-minimum-time control with *T*=0.25*s*

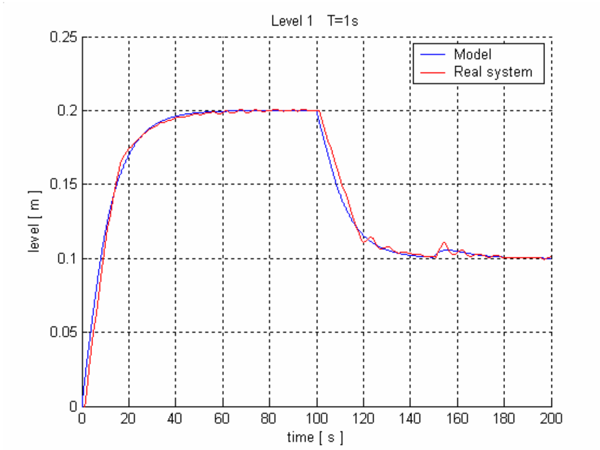

Fig.13 System output in non-minimum-time control with *T*=1*s*

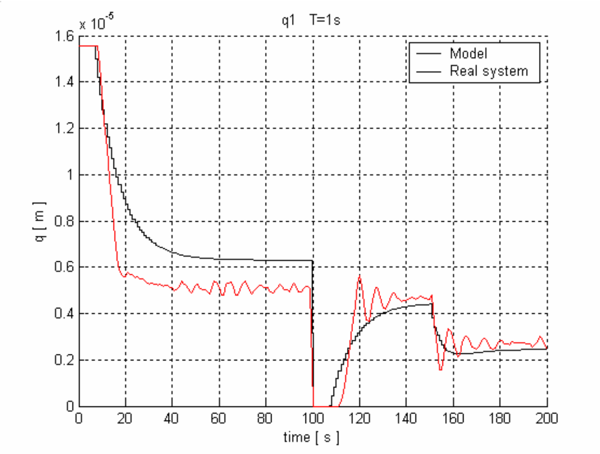

Fig.14 Controller output in non-minimum-time control with *T*=1*s*

Conclusions

In this paper the transfer function formalism in nonlinear control systems was employed to design various types of nonlinear discrete-time controllers for the real fluid tank system. We focused our attention on the controllers which linearize the system, eliminate the input disturbance and deals with the control signal constraint. The results are, in principle, similar to the linear theory. However, the remarkable difference is given by the fact that we are usually not able to find nonlinear discrete-time models of continuous-time systems, due to the nonlinear relations in systems equations. This has major consequences. Mainly, that quality of the designed nonlinear discrete-time controllers highly depend on the chosen sampling period *T*, as was demonstrated. All controllers, with various sampling periods, were implemented to control the real fluid tank system. We can conclude that to control the system satisfactorily the critical value of the sampling period *T* is about 1*s*.

Acknowledgments

The work was partially supported by: ESF Project 13120200015 Human Resource Development in Automation and Control, KEGA Project 3/3121/05 Network of Virtual Laboratories for Real Process Control, VEGA Project 1/3089/06 Development and Integration of Methods of the Nonlinear System Theory.

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