# DESIGN OF OBSERVERS FOR A CLASS OF NONLINEAR SYSTEMS IN ASSOCIATIVE OBSERVER FORM

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**Abstract.**

Conditions for the existence of an observer form for nonlinear discretetime dynamic models are known to be restrictive, motivating various extensions (e.g., generalized observer forms) to enlarge the class of systems for which observers with linear error dynamics can be designed. This paper introduces an alternative approach, based on replacing the usual addition operation + with a more general binary operation $\circ$ that is associative, continuous, and cancellative. These requirements lead to a simple representation for the operation $\circ$ in terms of a continuous, strictly monotonic function $\phi(\cdot)$ . This form is called an associative observer form, and it is demonstrated that the known results for observer design extend easily to this class of nonlinear systems and yield linear error dynamics. A constructive algorithm is described that determines whether the original nonlinear system can be transformed into the associative observer form. The proposed approach is compared with the generalized observer approach involving both state and output transformations, and it is shown that both approaches yield identical results. On the other hand, our approach simplifies the computations of the output transformation, which are done in two independent steps and do not require the solution of a system of n differential equations, as the generalized observer approach does.

**Keywords:** Nonlinear control system, Discrete-time system, Generalized observer form, Associative binary operation.

## INTRODUCTION

Consider the nonlinear system

$$\begin{aligned} x^+ &= G(x,u), \\ y &= h(x), \end{aligned} \qquad (1)$$

where $x \in \psi^{\varphi}$ is the state, $u, y \in \psi$ are respectively the input and output of the system. In this paper we consider the design of observers for discrete-time nonlinear systems of the form (1) by means of the so-called associative observer form which is a generalization of the standard observer form. Roughly speaking, a system in observer form is a linear observable system that is interconnected with an output and input-dependent nonlinearity:

$$\begin{aligned} z_1^+ &= z_2 + \varphi_1(z_1,u) \\ &\vdots \\ z_{n-1}^+ &= z_n + \varphi_{n-1}(z_1,u) \\ z_n^+ &= \varphi_n(z_1,u) \\ y &= z_1 \end{aligned} \qquad (2)$$

Observers for this kind of systems may be constructed by building a classical linear Luenberger observer for the linear part and adding the measurement-dependent non-linearity to this observer. The problem of transforming the discretetime nonlinear system (1) into the observer form (2) has been studied for systems with one output and without inputs in [9][1] and [3]. An extension to systems with inputs and a nonlinear output function *h(x)* is given by Ingenbleek [7]. Unfortunately, the conditions for the existence of an observer form are extremely restrictive. Therefore, different kind of generalizations have been considered to enlarge the class of systems for which one can design an observer with linear error dynamics: either the class of transformations allowed was enlarged or generalized observer forms were introduced [10]. For example, besides state transformations, also output transformations [6], system immersion into an higher dimensional system [8] or output-dependent time scale transformations [5] were considered. Finally, the paper [11] addresses the problem of transforming the nonlinear system into nonlinear observer canonical form in the extended state-space by augmenting the original system with some auxiliary states and defining virtual outputs. As a generalization of the observer forms, the so-called generalized output injection was introduced that, besides the outputs and the inputs, depends also on a finite number of their time derivatives (or shifts in the disctere time case).

This paper introduces an alternative generalization of the familiar observer form. The generalization presented here is based on replacing addition in observer form with a more general binary operation $\circ$ required only to be associative, continuous, and cancellative. These requirements then lead to a useful, simple representation for the operation $\circ$ in terms of continuous, strictly monotonic function $\phi(\cdot)$. This form is called an associative observer canonical form and our task is observer design for such systems. Our motivation is to explore the extent to which known results for observer design do or do not extend to this class of nonlinear systems.

[1] In [9], the order of the state variables is permuted.

## 1. ASSOCIATIVE BINARY OPERATORS

The binary operators $\circ$ considered here may be viewed as a mapping from some domain $D = I \times I$ into *I*, where *I* is an interval of real numbers that may be finite or infinite but must be open on at least one side. Further, $\circ$ is *associative* if it satisfies

$$(x \circ y) \circ z = x \circ (y \circ z) \tag{3}$$

for all *x, y* and *z* in *I.* Equivalently, this binary operation may be written as $x \circ y = F(x, y)$, reducing (3) to the *associativity equation*, [1]:

$$F[F(x, y), z] = F[x, F(y, z)],$$

for all $x, y, x \in I$. Further, $\circ$ is *continuous* if the map $F : I \times I \to I$ is continuous, and *cancellative* if either of the following conditions implies $t_1 = t_2 : t_1 \circ z = t_2 \circ z$ or $z \circ t_1 = z \circ t_2$. It has been shown in [1] that the binary operator $\circ$ is continuous, associative, and cancellative on *I* if and only if

$$x \circ y = \phi^{-1}[\phi(x) + \phi(y)], \tag{4}$$

where $\phi(\cdot)$ is strictly monotonic and continuous on *I*. The most common examples are addition, corresponding to $\phi(x) = x$, and multiplication, corresponding to $\phi(x) = \ln x$; the other examples are the *parallel combination* $x \parallel y$, defined as

$$x \parallel y = \frac{xy}{x+y},$$

arising from the parallel combination of resistances in electrical networks and defined by the function $\phi(x) = \frac{1}{x}$, and the projective addition operation $\oplus$ defined as

$$x \oplus y = \frac{2xy - (x+y)}{xy-1}$$

in [13] which corresponds to the function $\phi(x) = x/(x-1)$.

For convenience, the class of all associative, continuous and cancellative binary operators $\circ$ will be denoted $\Pi$. It follows from (4) that any binary operator $\circ$ $\Pi$ is also *commutative*: $x \circ y = y \circ x$, and, as a consequence, the combination:

$$\bigoplus_{i=1}^{n} x_i = x_1 \circ x_2 \circ \ldots x_n = \phi^{-1}\left[\sum_{i=1}^{n} \phi(x_i)\right]$$

is invariant under arbitrary permutations of the *n* terms $x_i$.

Another extremely useful consequence of the representation (4) is that the binary operation $\circ$ is *invertible*, with an inverse operation $\Diamond$ given explicitly by:

$$x \Diamond y = \phi^{-1}[\phi(x) - \phi(y)]. \qquad (5)$$

It follows directly from (4) and (5) that $(x \circ y) \Diamond y = x$. When $\circ$ denotes addition or multiplication, the inverse operations of subtraction and division are well-known. As less obvious examples, notenthat the inverses of the parallel combination $x \parallel y$ and the projective addition operation $x \oplus y$ are given by:

$$x \perp y = \frac{xy}{y-x}, \quad x \otimes y = \frac{x-y}{xy-2y+1}.$$

## 2. OBSERVER DESIGN FOR SYSTEMS IN ASSOCIATIVE OBSERVER FORM

The associative observer form is defined by replacing all additions in the observer canonical form (2) with arbitrary binary operation $\circ$ from $\Pi$:

$$\begin{aligned} z_1^+ &= z_2 \circ \varphi_1(z_1, u) \\ &\vdots \\ z_{n-1}^+ &= z_n \circ \varphi_{n-1}(z_1, u) \\ z_n^+ &= \varphi_n(z_1, u) \\ y &= z_1 \end{aligned} \qquad (6)$$

Relate to system (6) the following dynamical system

$$\begin{aligned} \hat{z}_1^+ &= \hat{z}_2 \circ \varphi_1(z_1, u) \circ \phi^{-1}\left(k_1[\phi(\hat{z}_1) - \phi(z_1)]\right) \\ &\vdots \\ \hat{z}_{n-1}^+ &= \hat{z}_n + \varphi_{n-1}(z_1, u) \circ \phi^{-1}\left(k_{n-1}[\phi(\hat{z}_1) - \phi(z_1)]\right) \\ z_n^+ &= \varphi_n(z_1, u) \circ \phi^{-1}\left(k_n[\phi(\hat{z}_1) - \phi(z_1)]\right) \end{aligned} \qquad (7)$$

called observer, and define for $i = 1, \ldots, n$, the error term $e_i = \phi(\hat{z}_i) - \phi(z_i)$.

**Proposition**. Observer (7) guarantees for system (6) in associative observer form a linear error dynamics

$$\begin{aligned} e_1^+ &= e_2 + k_1 e_1 \\ &\vdots \\ e_{n-1}^+ &= e_n + k_{n-1} e_1 \\ e_n^+ &= k_n e_1. \end{aligned} \qquad (8)$$

*Proof*. By (4) we can rewrite equation (6) as follows

$$\begin{aligned}\phi(z_1^+) &= \phi(z_2)+\phi(\varphi_1(z_1,u))\\ &\vdots\\ \phi(z_{n-1}^+) &= \phi(z_n)+\phi(\varphi_{n-1}(z_1,u))\\ z_n^+ &= \varphi_n(z_1,u),\end{aligned}$$

where $\phi$ is a strictly monotonous function defined by the binary operator $\circ$. Analogously, by (4) and the observation that

$$x\circ\phi^{-1}(y)=\phi^{-1}(\phi(x)+y)$$

yielding

$$\phi(x\circ\phi^{-1}(y))=\phi(x)+y$$

we can rewrite equation (7) as follows

$$\begin{aligned}\phi(\hat z_1{}^+) &= \phi(\hat z_2)+\phi(\varphi_1(z_1,u))+\\ &\quad +k_1[\phi(\hat z_1)]-\phi(z_1)\\ &\vdots\\ \phi(\hat z_{n-1}{}^+) &= \phi(\hat z_n)+\phi(\varphi_{n-1}(z_1,u))+\\ &\quad +k_{n-1}[\phi(\hat z_1)-\phi(z_1)]\\ \phi(\hat z_n{}^+) &= \phi(\varphi_n(z_1,u))+k_n[\phi(\hat z_1)-\phi\end{aligned}$$

. (9)

The straightforward computation yields now (8) for error dynamics.

∎

Suppose $\phi(\cdot)$ satisfies the generalized homogeneity condition ([2], p. 345):

$$\phi(kx)=\psi(k)\phi(x)\ , \tag{10}$$

for all $k,x\in I$. Let $x=1$ and note that

$$\phi(k)=\psi(k)\phi(1)\ \Rightarrow\psi(K)=\frac{\phi(k)}{\phi(1)} \tag{11}$$

It follows from this observation that $\psi(x)$ satisfies *Cauchy's power equation*:

$$\psi(kx)=\frac{\phi(kx)}{\phi(1)}=\frac{\psi(k)\phi(x)}{\phi(1)}=\psi(k)\psi(x). \tag{12}$$

for which the only continuous solutions are known to be [2], p. 31:

$$\psi(x)\equiv 0,\ \psi(x)=|x|^\nu,\ \psi(x)=|x|^\nu\ \mathrm{sign}(x), \tag{13}$$

where $\nu$ is any real constant. It follows from this result and the monotonicity requirement on $\phi(\cdot)$ that this function must be of the form:

$$\phi(x)=\alpha\,|x|^\nu\ \mathrm{sign}(x)\,, \tag{14}$$

for some nonzero real $\alpha$ and some $\nu>0$. Note that under these conditions, it follows that:

$$\phi^{-1}(x)=\left|\frac{x}{\alpha}\right|^{1/\nu}\mathrm{sign}\left(\frac{x}{\alpha}\right) \tag{15}$$

and that

$$\phi^{-1}(kx)=\left[|k|^{1/\nu}\ \mathrm{sign}(k)\right]\phi^{-1}(x)\equiv\tilde k\,\phi^{-1}(x). \tag{16}$$

The advantage of these observations is that if $\phi(\cdot)$ satisfies these conditions, we can draw two conclusions. First, the difficult term appearing in the observer equation (7) above for $j=1,\dots,n$ becomes:

$$\begin{aligned}\phi^{-1}\left(k_j[\phi(\hat z_1)-\phi(z_1)]\right) &= \tilde k_j\phi^{-1}(\phi(\hat z_1)-\phi(z_1))\\ &= \tilde k_j[\hat z_1\Diamond z_1]\end{aligned} \tag{17}$$

so the observer equations for $j=1,\dots,n-1$ now have the form:

$$\hat z_j^+=\hat z_{j+1}\circ\varphi_j(z_1,u)\circ\tilde k_j[\hat z_1\Diamond z_1].$$

and

$$\hat z_n^+=\varphi_n(z_1,u)\circ\tilde k_n[\hat z_1\Diamond z_1].$$

The other advantage of this condition is that it implies that $(I,\circ,\cdot)$ forms an algebraic ring, where $\cdot$ represents ordinary scalar multiplication. The key lies in the fact that the generalized homogeneity condition is sufficient to imply that $\cdot$ is distributive over $\circ$, for the proof, see Appendix. We conjecture that this condition is also necessary, i.e. that $\phi(\cdot)$ satisfies the generalized homogeneity conditions iff $(I,\circ,\cdot)$ forms a ring. However, the proof is left for future research.

If system (1) does admit an associative observer form, an observer for system (1) may then be obtained by first constructing an observer (7) for the system in the associative observer form (6) in the new coordinates $z = T(x)$ and then letting $\hat{x}(t) = T^{-1}(\hat{z}(t))$ be the estimate of $x(t)$.

We thus see that observer design for system (1) is relatively easy when (1) can be transformed into associative observer canonical form. This raises the question under what conditions (1) can be put into associative observer form.

## 3. SYSTEM TRANSFORMATION INTO ASSOCIATIVE OBSERVER CANONICAL FORM

The crucial point in the construction of a nonlinear observer of the form (7) with linear error dynamics (8) is the transformability of the discrete-time nonlinear system (1) into associative observer form (6). In [12] input-output difference equations with associative dynamics

$$y_{t+n} = \phi^{-1}\left(\sum_{i=0}^{n-1} \phi\left(\varphi_{i+1}(y_{t+i}, u_{t+i})\right)\right). \tag{18}$$

were studied with respect to realizability/realization and it was shown that the associative models of the form (18) do have a classical state space realization in the associative observer form (6). What is especially important with regard to the topic of this paper is that once the associative structure of the input-output model corresponding to (1) is recognized, the state space model construction in a associative observer form (6) is direct, allowing a simple translation from inputoutput model (18) to state space model (6). So, our approach is to find the input-output equation, corresponding to the state equations (1) for which we want to construct the observer and check if this equation can be put into associative i/o form (18).

Certainly, it is not always easy to recognize the associative model structure in input-otput equation

$$y_{t+n} = f\left(y_t, \ldots, y_{t+n-1}, u_t, \ldots, u_{t+n-1}\right) \tag{19}$$

by simple inspection since it depends on existence of certain functions $\phi, \varphi_1, \ldots, \varphi_n$, not defined in advance. In ([12] an algorithm was given to check if a higher order i/o difference equation can be written in the associative form (18). This algorithm permits computation of the required functions $\varphi_i$, $i = 1,2,\ldots,n$ step by step whenever they exist. The algorithm is constructive up to integrating some one-forms which is very common in nonlinear setting. To make this paper self-sufficient, we recall this algorithm below.

**Algorithm.** Calculate for $i = 0,1,\ldots,n-1$

$$\omega_{t+i} = \frac{\partial f(\cdot)}{\partial y_{t+i}} dy_{t+i} + \frac{\partial f(\cdot)}{\partial u_{t+i}} du_{t+i}. \tag{20}$$

Check:

$$d\omega_{t+i} \wedge \omega_{t+i} = 0. \tag{21}$$

If not, stop; otherwise

$$\omega_{t+i} = \lambda_{i+1}\left(y_t, \ldots, y_{t+n-1}, u_t, \ldots, u_{t+n-1}\right) d\varphi_{i+1}\left(y_{t+i}, u_{t+i}\right) \tag{22}$$

If (21) holds for all $i = 0,\ldots,n-1$, then according to formula (20) the total differential of function $f$,

$$df = \sum_{i=0}^{n-1} \omega_{t+i} = \sum_{i=0}^{n-1} \frac{\partial f}{\partial \varphi_{i+1}} d\varphi_{i+1} \tag{23}$$

can be written as

$$df = \sum_{i=0}^{n-1} \lambda_{i+1} d\varphi_{i+1} . \tag{24}$$

Consequently, for all $i = 0,\ldots,n-1$,

$$\lambda_{i+1} = \frac{\partial f}{\partial \varphi_{i+1}} . \tag{25}$$

However the problem of finding the function $\phi$ that determines the associative operator is still an open question. Though a complete solution remains a subject for future research, we suggest the following approach.

**Problem**. Given i/o equation (19) which is known to admit the associative i/o form (18) and the functions $\varphi_{i+1}$ find the function $\phi$ in (18).

**Solution.** From

$$\varphi\left[f\left(y_t,\ldots,y_{t+n-1},u_t,\ldots,u_{t+n-1}\right)\right] = \phi\left(\varphi_1\left(y_t,u_t\right)\right)+\ldots+\phi\left(\varphi_n\left(y_{t+n-1},u_{t+n-}\right.\right. \tag{27}$$

one can easily compute the following expressions

$$\frac{\phi'\left(\varphi_{i+1}\right)}{\phi'\left(\varphi_{j+1}\right)} = \frac{\lambda_{i+1}}{\lambda_{j+1}} . \tag{28}$$

for $i, j = 0,\ldots,n-1$. That is, from (27) and (20)

$$\phi'(f)\omega_{t+i} = \phi'(\varphi_{i+1})d\varphi_{i+1}$$

and now by (22)

$$\frac{\phi'(\varphi_{i+1})d\varphi_{i+1}}{\phi'(\varphi_{j+1})d\varphi_{j+1}} = \frac{\lambda_{i+1}d\varphi_{i+1}}{\lambda_{j+1}d\varphi_{j+1}}$$

yielding (28).

From equations (28) it is often possible to find the function $\phi(x)$. We will demonstrate the computations in Section 6 on three examples.

Unfortunately, contrary to what was claimed in [12], conditions (21) are not sufficient to transform the input-output equation (26) into the form (18), if $n > 2$.

The following system provides a counterexample

$$y_{t+3} = \frac{y_t y_{t+1} y_{t+2} u_{t+2}}{y_{t+1} + y_t}$$

for which the necessary condition (21) is satisfied, but the equation can not be written in the form (18). According to (20), the 1-forms $\omega_{t+i}$ are

$$\begin{aligned}
\omega_t &= \frac{y_{t+1}^2 y_{t+2} u_{t+2}}{\left(y_{t+1}+y_t\right)^2} dy_t , \\
\omega_{t+1} &= \frac{y_t^2 y_{t+2} u_{t+2}}{\left(y_{t+1}+y_t\right)^2} dy_{t+1} , \\
\omega_{t+2} &= \frac{y_t y_{t+1}}{\left(y_{t+1}+y_t\right)} dy_{t+2} u_{t+2} ,
\end{aligned} .$$

and by (21), the functions $\varphi_{i+1}$ and the integrating factors are, respectively as

$$\varphi_1 = y_t , \varphi_2 = y_{t+1} , \varphi_3 = y_{t+2} u_{t+2}$$

and

$$\begin{aligned}
\lambda_1 &= \frac{y_{t+1}^2 y_{t+2} u_{t+2}}{\left(y_{t+1}+y_t\right)^2} , \\
\lambda_2 &= \frac{y_t^2 y_{t+2} u_{t+2}}{\left(y_{t+1}+y_t\right)^2} , \\
\lambda_3 &= \frac{y_t y_{t+1}}{\left(y_{t+1}+y_t\right)} .
\end{aligned}$$

Compute

$$
\begin{aligned}
d\omega_{t+1} &= \frac{2y_t y_{t+1} y_{t+2} u_{t+2}}{(y_t + y_{t+1})^3} dy_{t+1} \wedge dy_t \; + \\
&+ \frac{y_{t+1}^2}{(y_t + y_{t+1})^2} d(y_{t+2}u_{t+2}) \wedge d \\
d\omega_{t+2} &= \frac{2y_t y_{t+1} y_{t+2} u_{t+2}}{(y_t + y_{t+1})^3} dy_t \wedge dy_{t+1} \; + \\
&+ \frac{y_t^2}{(y_t + y_{t+1})^2} d(y_{t+2}u_{t+2}) \wedge d \\
d\omega_{t+3} &= -\frac{y_{t+1}^2}{(y_t + y_{t+1})^2} d(y_{t+2}u_{t+2}) \wedge d \\
&- \frac{y_t^2}{(y_t + y_{t+1})^2} d(y_{t+2}u_{t+2}) \wedge d
\end{aligned} \tag{29}
$$

The direct computation shows that for $i = 0,1,2$

$$d\omega_{t+i} \wedge \omega_{t+i} = 0.$$

However, if we try to find $\phi$, we obtain the following relations

$$
\begin{aligned}
\frac{\lambda_1}{\lambda_2} &= \frac{y_{t+1}^2}{y_t^2}, \\
\frac{\lambda_1}{\lambda_3} &= \frac{y_{t+1} y_{t+2} u_{t+2}}{y_t (y_t + y_{t+1})}, \\
\frac{\lambda_3}{\lambda_3} &= \frac{y_t y_{t+2} u_{t+2}}{y_{t+1}(y_t + y_{t+1})}.
\end{aligned} \tag{30}
$$

The second formula in (30) leads to

$$\frac{\phi'(y_t)}{\phi'(y_{t+2}u_{t+2})} = \frac{\lambda_1}{\lambda_3} = \frac{y_{t+1} y_{t+2} u_{t+2}}{y_t (y_t + y_{t+1})}. \tag{31}$$

and since it contains a variable $y_{t+1}$, there does not exist $\phi$ as a single-variable function. The same happens with the third formula in system (30).

Consequently, the function: $\phi : \Psi \to \Psi$ does not exist for this example.

Really, if (21) holds for $i = 0,\dots,n-1$, then due to

$$df = \sum_{i=0}^{n-1} \omega_{t+i} = \sum_{i=0}^{n-1} \frac{\partial f}{\partial \varphi_{i+1}} d\varphi_{i+1} = \sum_{i=0}^{n-1} \lambda_{i+1} d\varphi_{i+1} \tag{32}$$

function $f$ can be written as a composite function (but not yet as in the form (18))

$$
\begin{aligned}
y_{t+n} &= f(y_t,\dots,y_{t+n-1},u_t,\dots,u_{t+n-1}) = \\
&= \varsigma(\varphi_1(y_t u_t),\dots,\varphi_n(y_{t+n-1}u_{t+n-1}))
\end{aligned}
$$

and the integrating factors $\lambda_{i+1}$ can be expressed as composite functions as well

$$\lambda(\varphi_1(y_t u_t),\dots,\varphi_n(y_{t+n-1}u_{t+n-1}). \tag{33}$$

Our next task is to find the necessary and sufficient conditions (see Theorem below) to transform the input-output equation into the form (18).

We start by proving a lemma that is fairly straightforward extension of Huijberts result [7], Theorem 6, $(i) \to (ii)$). Note however, that the associative dynamics form is different from the structure of the output equation considered in [6]. This lemma will be useful in proving the Theorem 2 below.

**Lemma.** If the input-output equation can be transformed into the associative dynamics form (18), there exists a function $S$ such that for $i = 0,\dots,n-1$

$$d\omega_{t+i} = dS \wedge \omega_{t+i}. \tag{34}$$

*Proof.* By (18),

$$\phi(f) = \sum_{i=0}^{n-1} \phi(\varphi_{i+1}(y_{t+i},u_{t+i})). \tag{35}$$

Taking the differential yields by (23)

$$\phi'(f)df = \sum_{i=0}^{n-1} \phi'(f)\omega_{t+1} = \sum_{i=0}^{n-1} \phi'(\varphi_{i+1})d\varphi_{i+1}. \tag{36}$$

Consequently, for $i = 0,1,\dots,n-1$

$$\phi'(f)\omega_{t+1} = \phi'(\varphi_{i+1})d\varphi_{i+1}. \quad (37)$$

Since the right hand side of (37) is a total differential, also the left hand side has to be a total differential and its exterior differential equals zero,

$$d[\phi'(f)] \wedge \omega_{t+i} + \phi'(f)d\omega_{t+i} = 0. \quad (38)$$

From (38), for $i = 0,\dots,n-1$

$$d\omega_{t+i} = -d\ln|\phi'(f)| \wedge \omega_{t+i} = dS \wedge \omega_{t+i}. \quad (33)$$

■

**Theorem 2.** The conditions

$$d\omega_{t+i} \wedge \omega_{t+j} + d\omega_{t+j} \wedge \omega_{t+i} = 0 \quad (40)$$
$$i, j = 0,\dots,n-1$$

are necessary and sufficient to transform the input-output equation (26) into the associative dynamics form (18).

*Proof.* Sufficiency. The proof falls naturally into three steps. On the first step (i) we will show that under the conditions of Theorem there exists $\Theta$, being a total differential of a certain function $S$ such that (34) holds. On the second step (ii) we will prove that (34) yields

$$\phi(f) = \sum_{i=0}^{n-1} \psi_{i+1}(y_{t+i}, u_{t+i}).$$

and finally, on the last step (iii) we will show that

$$\psi_{i+1} = \phi(\varphi_{i+1})$$

(i) First note that in case $i = j$ (40) yields (21). Taking the exterior differential of (22) we obtain

$$d\omega_{t+i} = d\ln|\lambda_{i+1}| \wedge \omega_{t+i}. \quad (41)$$

Obviously, $\Theta$ cannot be taken equal to $d\ln|\lambda_{i+1}|$ since there is no reason to assume that all integrating factors $\lambda_{i+1}$ are equal. However, we may search $\Theta = \Theta_{i+1} = \Theta_{j+1}\ \forall i, j$ in the form

$$\Theta_{i+1} = d\ln|\lambda_{i+1}| \wedge A_{i+1}\omega_{t+i}. \quad (42)$$

Then we have

$$d\omega_{t+i} = \Theta \wedge \omega_{t+i}. \quad (43)$$

For (43) to hold, we have to prove that under (40),

$$\Theta_{i+1} = \Theta_{j+1} \stackrel{\Delta}{=} \Theta \quad (44)$$

for all $i, j = 0,\dots,n-1$. Since the number of coordinates $\{y_t,\dots,y_{t+n-1}, u_t,\dots,u_{t+n-1}\}$ in the i/o space is $2n$ and the number of 1-forms $\omega_{t+k}$, $k = 0,\dots,n-1$, is $n$, they do not form the basis and every 1-form cannot be written as the linear combination of the 1-forms $\omega_t,\dots,\omega_{t+n-1}$. However, as we will show in the sequel, if (21) holds, the 1-forms $\Theta_{i+1}$ can be expressed as the linear combinations of $\omega_t,\dots,\omega_{t+n-1}$. By (42), (33) and (22)

$$\begin{aligned}
\Theta_{i+1} &= \frac{d\lambda_{i+1}}{\lambda_{i+1}} + A_{i+1}\omega_{t+i} \\
&= \frac{1}{\lambda_{i+1}} \sum_{k=0}^{n-1} \left( \frac{\partial\lambda_{i+1}}{\partial\varphi_{k+1}} d\varphi_{k+1} \right) + \\
&\quad + A_{i+1}\omega_{t+i} = \\
&= \frac{1}{\lambda_{i+1}} \sum_{k=0}^{n-1} \left( \frac{\partial\lambda_{i+1}}{\partial\varphi_{k+1}} \frac{1}{\lambda_{k+1}} \omega_{t+k} \right) + \\
&\quad + A_{i+1}\omega_{t+i} = \\
&\stackrel{\Delta}{=} \sum_{k=0}^{n-1} \vartheta_{i+1,t+k}\omega_{t+k}.
\end{aligned} \quad (45)$$

where

$$\vartheta_{i+1,t+k} = \frac{\partial\lambda_{i+1}}{\partial\varphi_{k+1}} \frac{1}{\lambda_{k+1}} \frac{1}{\lambda_{i+1}} + \delta_{ik} A_{k+i}$$

By substituting $\Theta_{i+1}$ from (45) into (43) we get

$$d\omega_{t+i} = \sum_{k=0}^{n-1} \vartheta_{i+1,t+k}\omega_{t+k} \wedge \omega_{t+i}$$

and substituting the last result into (40) we get for all $i, j = 0,\dots,n-1$

$$\sum_{k=0}^{n-1}\left(\vartheta_{i+1,t+k}-\vartheta_{j+1,t+k}\right)\omega_{t+k}\wedge\omega_{t+i}\wedge\omega_{t+j}=0$$

which yields (44).

To conclude the first step, note that from (42),

$$A_{i+1}\omega_{t+i}-A_{j+1}\omega_{t+j}=d\ln\left|\frac{\lambda_{j+1}}{\lambda_{i+1}}\right|$$

follows immediately that $A_{i+1}\omega_{t+i}$, for $i=0,\dots,n-1$ are total differentials. Therefore, $\Theta$ is also total differential and can be written as $\Theta = dS$.

(ii) Since by (23), $\Sigma_{i=0}^{n-1}\omega_{t+i}$ is a total differential, its exterior derivative

$$\sum_{i=0}^{n-1}d\omega_{t+i}=dS\wedge\sum_{i=0}^{n-1}\omega_{t+i}=dS\wedge df$$

equals zero and by Cartan's Lemma, $dS\in\operatorname{span}\{df\}$. Therefore, a function $S(y_t,\dots,y_{t+n-1},u_t,\dots,u_{t+n-1})$ can be expres-sed as a composite function, $S=T\circ f$. The latter allows us to define a function $\phi$ such that

$$\phi'(f)=e^{-S}. \tag{46}$$

Then

$$S=-\ln|\phi'(f)|. \tag{47}$$

According to (34) and (46), for $i=0,\dots,n-1$

$$d\omega_{t+i}=-d\ln|\phi'(f)|\wedge\omega_{t+i} \tag{48}$$

from where by direct computation we get

$$d\left[\phi'(f)\omega_{t+i}\right]=0. \tag{49}$$

Therefore, there exist functions $\psi_{i+1}(y_{t+i},u_{t+i})$ such that

$$\phi'(f)\omega_{t+i}=d\psi_{i+1}. \tag{50}$$

Multiplication of (23) by $\phi'(f)$ gives

$$d[\phi(f)]=\sum_{i=0}^{n-1}\phi'(f)\omega_{t+i}=\sum_{i=0}^{n-1}d\psi_{i+1}(y_{t+i},u_t \tag{51}$$

yielding

$$\phi(f)=\sum_{i=0}^{n-1}\psi_{i+1}(y_{t+i},u_{t+i}) \tag{52}$$

(iii) Finally, we have to show, that for $i=0,\dots,n-1$

$$\psi_{i+1}=\phi(\varphi_{i+1}). \tag{53}$$

Due to (50), (22) and (25)

$$\begin{aligned} d\psi_{i+1} &= \phi'(f)\lambda_{i+1}d\varphi_{i+1} = \\ &= \phi'(f)\frac{\partial f}{\partial\varphi_{i+1}}d\varphi_{i+1} = \\ &= \phi'(\varphi_{i+1})d\varphi_{i+1}\end{aligned} \tag{54}$$

yielding

$$\psi_{i+1}=\phi(\varphi_{i+1}). \tag{55}$$

*Necessity*. Denote

$$\sum_{i=0}^{n-1}\phi(\varphi_{i+1}(y_{t+i},u_{t+i}))=\chi.$$

Then

$$\omega_{t+i}=\lambda_{i+1}d\varphi_{i+1}$$

where

$$\lambda_{i+1}(\phi^{-1})|_\chi\cdot\phi'(\varphi_{i+1}) \tag{56}$$

and

$$\begin{aligned} d\omega_{t+i} &= d\lambda_{i+1}\wedge d\varphi_{i+1} \\ &= d\ln|\lambda_{i+1}|\wedge\omega_{t+i}\end{aligned}. \tag{57}$$

Direct computation now gives

$$\begin{aligned} &d\omega_{t+i}\wedge\omega_{t+j}+d\omega_{t+j}\wedge\omega_{t+i}= \\ &=d\ln\left|\frac{\lambda_{i+1}}{\lambda_{j+1}}\right|\wedge\omega_{t+i}\wedge\omega_{t+j}.\end{aligned} \tag{58}$$

From (56)

$$\begin{aligned} &d\ln\left|\frac{\lambda_{i+1}}{\lambda_{j+1}}\right|=d\ln\left|\frac{\phi'(\varphi_{i+1})}{\phi'(\varphi_{j+1})}\right|= \\ &=\frac{\phi'(\varphi_{j+1})}{\phi'(\varphi_{i+1})}d\left[\frac{\phi'(\varphi_{i+1})}{\phi'(\varphi_{j+1})}\right]\end{aligned},$$

where

$$d\left[\frac{\phi'(\varphi_{i+1})}{\phi'(\varphi_{j+1})}\right] = \frac{1}{[\phi'(\varphi_{j+1})]^2}\left[d(\phi'(\varphi_{i+1}))\phi'(\varphi_{j+1}) - d(\phi'(\varphi_{j+1}))\phi'(\varphi_{i+1})\right]. \quad (59)$$

Substituting (59) into (43) and taking into account that

$$d\phi'(\varphi_{i+1}) = \phi''(\varphi_{i+1})d\varphi_{i+1} = \frac{\phi''(\varphi_{i+1})}{\lambda_{i+1}}\omega_{t+i}$$

(40) follows immediately.

∎

*Remark*. In the case $n=2$ condition (21) is both necessary and sufficient. It follows from the fact that under (21), $df = \omega_t + \omega_{t+i}$ and then $d\omega_t = -d\omega_t$, yielding (40).

## 4. COMPARISON OF TWO METHODS

In this section we compare the method used to transform the input-output difference equation (26) into the associative observer canonical form, described in the present paper, with the method given in [6]. Note that Huijberts considers the equations

$$y_{t+n} = f(y_t, \ldots, y_{t+n-1}) \quad (60)$$

without inputs and searches for the output transformation $p : \psi \to \psi$, so that the function $f$ in (60) satisfies

$$p(f) = \phi_n(y_t) + \ldots + \phi_1(y_{t+n-1})$$

The corresponding step in our method is to transform equation (60) into the form (18). Since (60) does not contain the inputs, we require

$$\phi(y_{t+n)}) = \phi(\varphi_1(y_t)) + \ldots + \phi(\varphi_n(y_{t+n-1})). \quad (62)$$

Comparison of (61) and (62) demonstrates the analogy between two methods, and gives

$$\phi = p,\ \phi(\varphi_i) = \phi_{n-i+1},\ i = 1,\ldots,n. \quad (63)$$

To determine the function $p$, Huijberts defines the 1-forms

$$\grave{\omega}_i = \sum_{j=1}^{i}\frac{\partial f}{\partial y_{t+j-1}}dy_{t+j-1}. \quad (64)$$

The corresponding step in our method is formula (20) to define the 1-forms

$$\omega_{t+i-1} = \frac{\partial f}{\partial y_{t+i-1}}dy_{t+i-1}. \quad (65)$$

Obviously, the 1-forms $\omega_{t+1-i}$ in our method and the 1-forms $\grave{\omega}_i$ used in the Huijberts' method are related in the following way

$$\omega_{t+i-1} = \grave{\omega}_i - \grave{\omega}_{i-1}. \quad (66)$$

In order to transform (60) into the form

$$y_{t+n} = p^{-1}\left(\phi_n(y_t) + \ldots + \phi_1(y_{t+n-1})\right)$$

all the 1-forms

$$\tilde{\omega}_i = \sum_{j=1}^{n}\frac{\partial(p(f))}{\partial y_{t+j-1}}dy_{t+j-1} = (p'(f))\grave{\omega}_i$$

for $i = 1,\ldots,n$ have to be total differentials, therefore, for $i = 1,\ldots,n$, the following has to hold:

$$d(p'(f)) \wedge \grave{\omega}_i + (p'(f))d\grave{\omega}_i = 0.$$

The latter yields the system of differential equations to determine the function $p$:

$$d\grave{\omega}_i = -d\ln|p'(f)| \wedge \grave{\omega}_i. \quad (67)$$

The 1-forms $\grave{\omega}_i$ are in general not exact, but according to [6], the multiplication of $\grave{\omega}_i$ with the function $p'(f)$ gives us the exact 1-forms

$$\tilde{\omega}_i = (p'(f))\grave{\omega}_i. \quad (68)$$

Using (67), one can find the function

$$p'(x)|_{x=f(y_t,\ldots,y_{t+n-1})}$$

and then, via integrating, $p(x)$, but both steps are not easy taks, in general.

Since $(p'(f))\grave{\omega}_i$ have to be total diffe-rentials for $i = 1,\ldots,n$, the same has to hold for

$$(p'(f))[\omega_i - \omega_{i-1}] = (p'(f))\omega_{t+i-1}$$
(see 66)).

According to formulae (66) and (67), one may also write

$$d\omega_{t+i-1} = -d\ln(p'(f)) \wedge \omega_{t+i-1} . \quad (69)$$

Using our method, the corresponding step is to check for $i = 1,\dots,n$ the condition

$$d\omega_{t+i-1} \wedge \omega_{t+i-1} = 0 \quad (70)$$

which, if satisfied, yields

$$\omega_{t+i-1} = \lambda_i(y_t,\dots,y_{t+n-1})d\varphi_i(y_{t+i-1}) . \quad (71)$$

The latter means that by multiplying the 1-forms $\omega_{t+i-1}$ by the integrating factor $\lambda_i^{-1}(y_t,\dots,y_{t+n-1})$ gives us also the exact differentials.

Next we look for the relationship between the coefficients $\lambda_i$ in (22) and $p'(f)$ in (68). According to (62) one can write equation (60) also in the form

$$p(f) = p(\varphi_1(y_t)) + \dots + p(\varphi_n(y_{t+n-1}))$$

Taking the partial derivative with respect to $y_{t+i}$ yields according to (64)

$$p'(f)\omega_{t+i} = p'(\varphi_{i+1})d\varphi_{i+1} .$$

Comparing the obtained result with (71) we get

$$\lambda_{i+1} = \frac{p'(\varphi_{i+1})}{p'(f)} . \quad (72)$$

Of course, one can also find the different integrating factors.

Taking into account that $p(x) = \phi(x)$, one can find $p(x)$ from the system of equations (28).

## 5. EXAMPLES

### Example 1

Consider the output equation

$$y_{t+2} = \frac{2y_t y_{t+1} - y_t - y_{t+1}}{y_t y_{t+1} - 1} .$$

To calculate the 1-forms (20), we take the partial derivatives

$$\frac{\partial f}{\partial y_t} = \frac{(y_{t+1}-1)^2}{(y_t y_{t+1}-1)^2}$$

and

$$\frac{\partial f}{\partial y_{t+1}} = \frac{(y_t-1)^2}{(y_t y_{t+1}-1)^2}$$

to obtain

$$\omega_t = \frac{\partial f}{\partial y_t}dy_t , \qquad \omega_{t+1} = \frac{\partial f}{\partial y_{t+1}}dy_{t+1}$$

The coefficients $\lambda_1$ and $\lambda_2$ in (22) are

$$\lambda_1 = \frac{(y_{t+1}-1)^2}{(y_t y_{t+1}-1)^2} , \quad \lambda_2 = \frac{(y_t-1)^2}{(y_t y_{t+1}-1)^2}$$

and we finally obtain that $\varphi_1(y_t) = y_t$ and

$$\varphi_2(y_{t+1}) = y_{t+1} .$$

To compute $\phi$, by (28)

$$\frac{\phi'(\varphi_1)}{\phi'(\varphi_2)} = \frac{(y_{t+1}-1)^2}{(y_t-1)^2}$$

yielding

$$\phi'(x) = \frac{1}{(x-1)^2} .$$

One can find $\phi(x)$ as

$$\phi(x) = -\int \frac{dx}{(x-1)^2} = \frac{x}{x-1} . \quad (73)$$

This choice will yield

$$\phi(y_t) + \phi(y_{t+1}) = \frac{y_t}{y_t-1} + \frac{y_{t+1}}{y_{t+1}-1} = \\ = \frac{2y_t y_{t+1} - y_t - y_{t+1}}{y_t y_{t+1} - y_t - y_{t+1} - 1} := Y$$

and since

$$\phi^{-1}(\zeta) = \frac{\zeta}{\zeta - 1} ,$$

$$\phi^{-1}\left[\phi(y_t)+\phi(y_{t+1})\right]=\frac{Y}{Y-1}=$$
$$=\frac{2y_t y_{t+1}-y_t-y_{t+1}}{y_t y_{t+1}-1}.$$

We can also calculate function $p(x)=\phi(x)$ using Huijberts' method. Calculating 1-forms $\dot{\omega}_i$ according to (64) we get

$$\dot{\omega}_1=\frac{\partial f}{\partial y_t}dy_t=\frac{(y_{t+1}-1)^2}{(y_t y_{t+1}-1)^2}dy_t,$$
$$\dot{\omega}_2=\frac{\partial f}{\partial y_t}dy_t+\frac{\partial f}{\partial y_{t+1}}dy_{t+1}=$$
$$=\frac{(y_{t+1}-1)^2}{(y_t y_{t+1}-1)^2}dy_t+\frac{(y_t-1)^2}{(y_t y_{t+1}-1)^2}dy_t \tag{74}$$

To find the function

$$S=-\ln(p'(f)), \tag{75}$$

necessary for calculating $p$, we use formula (67) and calculate the exterior differentials of $\dot{\omega}_1$ and $\dot{\omega}_2$. Then

$$d\dot{\omega}_1=\frac{\partial^2 f}{\partial y_t \partial y_{t+1}}dy_{t+1}\wedge dy_t=$$
$$=\frac{2(y_t-1)(y_{t+1}-1)}{(y_t y_{t+1}-1)^2}dy_{t+1}\wedge dy_t=$$
$$=\mathrm{dS}\wedge\dot{\omega}_1. \tag{76}$$

Consequently, if $S$ exists, by (74) and (76)

$$\mathrm{dS}\wedge\dot{\omega}_1=\frac{\partial S}{\partial y_{t+1}}dy_{t+1}\wedge\frac{(y_{t+1}-1)^2}{(y_t y_{t+1}-1)^2}dy_t$$

with

$$\frac{\partial S}{\partial y_{t+1}}=\frac{2(y_t-1)}{(y_{t+1}-1)(y_t y_{t+1}-1)^2}. \tag{77}$$

Since $\dot{\omega}_2$ is an exact 1-form, condition (67) gives

$$d\dot{\omega}_2=dS\wedge\dot{\omega}_2=0$$

yielding

$$\frac{\partial S}{\partial y_t}=\frac{2(y_{t+1}-1)}{(y_t-1)(y_t y_{t+1}-1)^2}. \tag{78}$$

The function $S$ can be found by solving the system of differential equations (77) and (78). Note that in the general case this can be a very complicated task. Taking into account also (75) we get

$$p'(f)=\frac{(y_t y_{t+1}-1)^2}{(y_t-1)^2(y_{t+1}-1)^2}. \tag{79}$$

To determine the function $p(x)$ the right hand side of (79) has to be expressed in terms of funtion $f$. This step, again ext-remely complicated, leads to the result

$$p'(f)=\frac{1}{(1-f)^2}$$

or

$$p'(x)=\frac{1}{(1-x)^2}. \tag{80}$$

The integration gives

$$p(x)=\frac{x}{1-x}, \tag{81}$$

the same result as in (73), obtained by our method.

## Example 2

Consider the i/o equation

$$y_{t+2}=u_t(bu_{t+1}+cy_{t+1}+ay_{t+1}u_{t+1}) \tag{82}$$

Compute

$$\omega_t=(bu_{t+1}+cy_{t+1}+ay_{t+1}u_{t+1})du_t,$$
$$\omega_{t+1}=u_t\left[(b+ay_{t+1})du_{t+1}+(c+au_{t+1})dy_{t+1}\right]. \tag{83}$$

from which

$$\varphi_1=u_t,$$
$$\varphi_2=bu_{t+1}+cy_{t+1}+ay_{t+1}u_{t+1}.$$

and $\lambda_1=\varphi_2$, $\lambda_2=\varphi_1$. Therefore, by (28)

$$\frac{\phi'(\varphi_1)}{\phi'(\varphi_2)} = \frac{\varphi_2}{\varphi_1}.$$

The latter yields

$$\phi'(x) = \frac{1}{x}$$

and

$$\phi(x) = \ln|x|.$$

Applying the Huijberts' method one computes

$$\begin{aligned} \dot\omega_1 &= \omega_t, \\ \dot\omega_2 &= \omega_t + \omega_{t+1} \end{aligned} \qquad (84)$$

and

$$\begin{aligned} d\dot\omega_1 &= d(bu_{t+1} + cy_{t+1} + ay_{t+1}u_{t+1}) \wedge du_t, \\ d\dot\omega_2 &= 0. \end{aligned}$$

So, $d\dot\omega_1 = dS \wedge \dot\omega_1$, and by taking into account i/o equation (82) and relationship (84), the latter yields

$$\begin{aligned} dS = {} & d\ln(bu_{t+1} + cy_{t+1} + ay_{t+1}u_{t+1}) + \\ & + q(y_t, y_{t+1}, u_t, u_{t+1})du_t. \end{aligned}$$

From here it is in general not possible to find the function $q$.

**Example 3**

Consider the i/o equation

$$y_{t+2} = \frac{au_t y_{t+1} u_{t+1}}{u_t + ay_{t+1}u_{t+1}}. \qquad (85)$$

Compute

$$\begin{aligned} \omega_t &= \frac{a^2 y_{t+1}^2 u_{t+1}^2}{(u_t + ay_{t+1}u_{t+1})^2} du_t, \\ \omega_{t+1} &= \frac{u_t^2}{(u_t + ay_{t+1}u_{t+1})^2} d(ay_{t+1}u_{t+1}), \end{aligned} \qquad (86)$$

from which

$$\begin{aligned} \varphi_1 &= u_t, \\ \varphi_2 &= ay_{t+1}u_{t+1} \end{aligned}$$

Applying now formula (28) we obtain

$$\frac{\phi'(\varphi_1)}{\phi'(\varphi_2)} = \frac{\varphi_2^2}{\varphi_1^2}.$$

From above we get

$$\phi'(x) = -\frac{1}{x^2}$$

and finally

$$\phi(x) = \frac{1}{x}. \qquad (87)$$

Applying the Huijberts' method one computes

$$\begin{aligned} \dot\omega_1 &= \omega_t, \\ \dot\omega_2 &= \omega_t + \omega_{t+1} \end{aligned} \qquad (88)$$

and

$$\begin{aligned} d\dot\omega_1 &= \frac{2au_t y_{t+1}u_{t+1}}{(u_t + ay_{t+1}u_{t+1})^3} d(ay_{t+1}u_{t+1}) \wedge du_t = \\ &= \mathrm{dS} \wedge \dot\omega_1 = \\ &= \frac{\partial S}{\partial(ay_{t+1}u_{t+1})} d(ay_{t+1}u_{t+1}) \wedge du_t, \\ d\dot\omega_2 &= 0. \end{aligned} \qquad (89)$$

Comparing formulas (88), (89) and (86) we get

$$\frac{\partial S}{\partial(ay_{t+1}u_{t+1})} = \frac{2u_t}{ay_{t+1}u_{t+1}(u_t + ay_{t+1}u_{t+1})}$$

yielding

$$\begin{aligned} S = -\ln|p'(f)| = \ln\left(\frac{ay_{t+1}u_{t+1}}{u_t + ay_{t+1}u_{t+1}}\right)^2 + \\ + \ln|g^2(u_t)|, \end{aligned}$$

where the last term is a function depending only on $u_t$. Then

$$p'(f) = \left[\frac{u_t + ay_{t+1}u_{t+1}}{ag(u_t)y_{t+1}u_{t+1}}\right]^2$$

.

Since the last expression has to be expressed via $f$ in (85), we take

$$g(u_t) = u_t$$
and get
$$p'(f) = \frac{1}{f^2},$$
yielding $p(x) = -1/x$. This solution agrees up to the sign with the solution for $\phi(x)$ in (87).

## 6. CONCLUSIONS

This paper has introduced a new class of nonlinear observers that exhibit linear error dynamics. The basis for this observer class is the replacement of the usual addition operator + with a more general operator $\circ$ in the canonical observer form for nonlinear discrete-time dynamic models that has been considered previously by a number of authors. The resulting structure, called the associative observer form, is significantly more flexible than the canonical observer form, greatly enlarging the class of nonlinear models that can be represented. The operator $\circ$ on which this extension is based still exhibits a number of characteristics of the usual addition operator: in particular, it is required to be associative, continuous, and cancellative, implying that it may be expressed in terms of a continuous, strictly monotonic function $\phi(\cdot)$. Taking $\phi(x) = x$ reduces $\circ$ to + and reduces the associative observer form introduced here to the well-known canonical observer form. Allowing $\phi(\cdot)$ to be more general but requiring it to satisfy the generalized homogeneity condition discussed in Sec. 3 implies that the operations $\circ$ and scalar multiplication still form a ring, as in the case where $\circ$ is the usual addition operator, and leads to an associative observer that corresponds to the usual linear observer but with slightly modified gains, with + replaced by $\circ$ and with − replaced by $\diamond$, the inverse of the $\circ$ operator, which is also simply expressed in terms of the function $\phi(\cdot)$. Relaxing the generalized homogeneity condition leads to the most general case, where the associative observer form is slightly more complex but still reasonably straightforward. In all cases, the error dynamics remain linear, as shown in Section 3.

In addition to defining the associative observer class just described, we have also presented a realization procedure for putting a given nonlinear discrete-time dynamic model into associative observer form. We demonstrate this method for three simple examples and also compare it with the method of Huijberts, which leads to the same result for two of these three examples but fails to yield a solution for the third. In addition, the computations involved in our procedure are substantially simpler.

## 7. APPENDIX

**Proposition.** Under the generalized homogeneity condition $(I, \circ, \cdot)$ forms an algebraic ring.

*Proof.* A ring (see [4],, p. 103), is defined as a set $X$ with two binary operations, $\circ$ and $\cdot$ that satisfy the following axioms:

A1: $\circ$ is commutative: $x \circ y = y \circ x$ for all $x, y \in X$;
A2: $\circ$ is associative: $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in X$;
A3: a zero element exists, $Z \in X$, such that $x \circ Z = x$ for all $x \in X$;
A4: additive inverses: for all $x \in X$, there exists $z \in X$ such that $x \circ z = Z$;

M1: the operation $\cdot$ is associative: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in X$;
D1: both operations satisfy left distributivity: $x \cdot (y \circ z) = x \cdot y \circ x \cdot z$;
D2: both operations satisfy right distributivity:
$(x \circ y) \cdot z = x \cdot z \circ y \cdot z$.

In our paper, the operation $\circ$ is defined to be associative and it is shown in Section 2 that it is also commutative. To show the existence of a zero element, note that $x \circ Z = x$ implies

$$\phi(x) + \phi(Z) = \phi(x) \Rightarrow \phi(Z) = 0 \Rightarrow Z = \phi^{-1}( \tag{90}$$

Further, since $\circ$ exhibits the inverse operation $\Diamond$, it follows that the additive inverse $z$ for any element $x \in X$ is simply $z = Z \Diamond x$. Thus, our operation _ always satisfies conditions A1 through A4.

In the case of interest here, the operation $\cdot$ is simply scalar multiplication, which is both associative and commutative, so that condition M1 is satisfied and conditions D1 and D2 are equivalent. Thus, the collection $(X, \circ, \cdot)$ defines a ring if and only if distributivity condition D1 is satisfied. In terms of the function $\phi(\cdot)$, this condition is:

$$\mathrm{x} \cdot \phi^{-1}(\phi(y) + \phi(z)) = \phi^{-1}(\phi(x \cdot y) + \phi(x \cdot z)) \tag{91}$$

Next, suppose $\phi(\cdot)$ satisfies the generalized homogeneity condition:

$$\phi(x \cdot y) = \psi(x) \cdot \phi(y), \tag{92}$$

and note that the right-hand side of Eq. (91) then reduces to

$$\phi^{-1}(\phi(x \cdot y) + \varphi(x \cdot z)) = \phi^{-1}(\psi(x) \cdot [\phi(y) + \varphi \tag{93}$$

Setting $v = \phi(y)$ and applying the inverse function $\phi^{-1}$ to both sides of Eq. (92) yields the result that

$$\phi^{-1}(\psi(x) \cdot v) = x \cdot \phi^{-1}(v). \tag{94}$$

Combining Eqs. (93) and (94) then gives

$$\phi^{-1}(\phi(x \cdot y) + \phi(x \cdot z)) = x \cdot \phi^{-1}(\phi(y) + \phi(z)). \tag{95}$$

establishing that distributivity condition D1 holds. Thus, $\phi(\cdot)$ satisfies the generalized homogeneity condition (92), it follows that $(X, \circ, \cdot)$ defines a ring.


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