Robust stability analysis using polynomially dependent Lyapunov function

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Abstract
Robust stability analysis of uncertain discrete time systems is studied. The LMI robust stability analysis method based on polynomial parameter dependent Lyapunov function is presented. This method is compared with other robust stability analysis methods formulated through LMI using linear parameter dependent Lyapunov function. The results are tested on randomly generated examples.

Keywords: robust stability, discrete-time systems, stability analysis, polytopic systems, linear matrix inequalities (LMI)

Introduction

Robust stability of uncertain dynamic systems has major importance when the real world system models are considered. The realistic approach includes uncertainties of various kinds into the system model and a basic required quality of the system is its stability in the whole uncertainty domain – this quality is called robust stability. A possible approach to consider system uncertainties is to use affine or polytopic description of uncertain system. Polytopic model is appropriate for using LMI approach to robust stability analysis and robust control design.

Recently developed Linear Matrix Inequality (LMI) approach belongs to the most attractive ones due to its computational efficiency – the respective interior-point based algorithm provides the solutions in polynomial time (e.g. Boyd *et al.* 1994). Therefore significant effort has been made to transform major control problems into LMI framework (e.g.: Boyd *et al.* 1994, Skelton *et al.* 1998, Dettori and Scherer 2000; Peaucelle, *et al.* 2000; Henrion, *et al.* 2002, deOliveira *et al.* 1999; Rosinová and Veselý 2004; Veselý 2007).

In robust control of linear systems a quadratic stability notion has been introduced, where one Lyapunov function is considered for the whole uncertainty domain. This approach includes robustness against arbitrarily quick changes of system parameters within the uncertainty domain, however for slowly varying systems it yields too conservative results. Therefore the parameter dependent Lyapunov function has been developed which enables to obtain less conservative results. Recently, more general, though computationally demanding form of Lyapunov function – a polynomially parameter-dependent one has been studied with promising results, (Bliman 2004, Montagner et al., 2006, Ebihara *et al*. 2006 and references therein).

In this paper several stability analysis methods for linear uncertain discrete-time systems are compared. Polynomially parameter-dependent Lyapunov function is presented, yielding the respective sufficient condition for linear discrete-time polytopic system. Qualities of the considered stability analysis methods are studied and compared on the random generated examples.

1. Robust stability for uncertain discrete-time system

This section is devoted to robust stability problem formulation for discrete-time linear systems; several recent results for robust stability analysis using LMI approach are recalled. Consider a linear discrete-time uncertain system with polytopic uncertainty domain:

$$x(k+1) = A(\alpha)x(k) \qquad (1)$$

where

$$A(\alpha) \in \left\{ \sum_{i=1}^{N} \alpha_i A_i, \quad \sum_{i=1}^{N} \alpha_i = 1, \quad \alpha_i \geq 0 \right\}, \qquad (2)$$

$x(k) \in R^n$ is state vector; A_i are known constant matrices of appropriate dimensions corresponding to the nominal system.

$\alpha \in R^N$ denotes a vector of α_i respective to (2).

The uncertain system (1), (2) can be equivalently described by vertices of the respective polytope

$$\{A_1, A_2, ..., A_N\}. \qquad (3)$$

The aim of robust stability analysis is to check the stability of the uncertain system (1), (2) or, equivalently, (3) in the whole uncertainty domain.

We start with basic notions. *Quadratic stability* corresponds to the existence of one Lyapunov function for the whole uncertainty domain as determined in the following definition.

Lemma 1 (Quadratic stability)

The polytopic system (1), (2) is *quadratically stable* if and only if there exists a symmetric positive definite matrix P such that

$$A(\alpha)^T P A(\alpha) - P < 0 \qquad (4)$$

Quadratic stability guarantees the stability in the whole uncertainty domain including arbitrary quick changes of elements of matrix $A(\alpha)$, however it is often overly conservative, when the slowly varying systems are considered. To reduce the conservatism of quadratic stability, the *parameter-dependent Lyapunov function* $P(\alpha)$ (denoted in the sequel as PDLF) has been introduced for uncertain system (1), (2). The respective robust stability notion is considered according to (deOliveira *et al*. 1999):

Definition 1

System (1) is *robustly stable* in the convex uncertainty domain (2) with parameter-dependent Lyapunov function if and only if there exists a matrix $P(\alpha)=P(\alpha)^T>0$ such that

$$A(\alpha)^T P(\alpha) A(\alpha) - P(\alpha) < 0 \tag{5}$$

for all α such that $A(\alpha)$ is given by (2).

For a polytopic uncertain system we will consider PDLF in a form

$$P(\alpha)=\sum_{i=1}^{N}\alpha_i P_i \ . \tag{6}$$

The following fundamental robust stability result decouples Lyapunov matrix from a system matrix enabling the use of PDLF in analysis and design.

Lemma 2 (deOliveira *et al.* 1999)

Uncertain system (1), is robustly stable in uncertainty domain (2) if there exist symmetric matrices P_i and a matrix G such that

$$\begin{bmatrix} -P_i & A_i^T G^T \\ GA_i & -G-G^T+P_i \end{bmatrix} < 0 \tag{7}$$

The robust stability condition in general form with two auxiliary matrices has been presented in (Dettori and Scherer, 2000) and (Peaucelle et all., 2000). The sufficient robust stability condition for a discrete-time system is:

Lemma 3 (Peaucelle *et al.* 2000)

Uncertain system (1) is robustly stable in uncertainty domain (2) if there exist symmetric matrices P_i and a matrices H and G such that

$$\begin{bmatrix} HA_i + A_i^T H^T - P_i & A_i^T G - H \\ G^T A_i - H^T & -G-G^T+P_i \end{bmatrix} < 0 \qquad i=1,2,\dots,N. \tag{8}$$

It is important to note that there are no specific requirements on matrices H and G.

Robust stability conditions from Lemma 2 and 3 consider parameter dependent Lyapunov function (6) linear in the uncertain parameters. Therefore the received stability conditions are sufficient only and there still remains space to relax them closer to necessary and sufficient ones. One possible way is to consider polynomial Lyapunov function.

In this paper we consider polynomially parameter-dependent Lyapunov function (PPDLF) according to (Ebihara *et al*. 2006):

$$P(\alpha)=G(M(\alpha),p)^T \Pi_p(\alpha) G(M(\alpha),p) \tag{9}$$

where

$M(\alpha)\in R^{n\times n}$ is a given affine function of scalar parameter α;

$$G(M(\alpha),p)=[I_n, M(\alpha)^T,\dots,\left(M(\alpha)^p\right)^T]^T \tag{10}$$

is a *given* polynomial matrix function of α, *p* denotes the degree of this polynomial with respect to α, in the sequel we will denote $G(M(\alpha),p)$ simply as $G(\alpha,p)$.

Note that for $M(\alpha)=0$, we have $G(0,p)=[I_n,0_n,\dots,0_n]^T$, $I_n\in R^{n\times n}$ is identity matrix and $0_n\in R^{n\times n}$ is zero matrix.

$\Pi_p(\alpha)$ is an affine matrix function of α, to be determined through optimization. For *nxn* system, $\Pi_p(\alpha)$ is (*p*+1)*n*x(*p*+1)*n* symmetric matrix.

The following denotation for orthogonal complement is used in this paper.

For a matrix $A\in R^{m\times n}$ with $rank(A)=r<n$, $A^{\perp}\in R^{m\times(n-r)}$ is a matrix such that $AA^{\perp}=0$ and $(A^{\perp})^T A^{\perp}>0$.

In the next developments the frequently used Finsler's lemma is used to obtain robust stability condition for polynomially parameter-dependent Lyapunov function in the form of LMI.

Lemma 4 (Skelton 1998, Ebihara *et al.* 2006)

Let matrices $Q\in R^{n\times n}$ and $B\in R^{m\times n}$ be given such that $rank(B)<n$. Then the following conditions are equivalent.

(i) The inequality $(B^{\perp})^T QB^{\perp}<0$ holds.

(ii) There exists $\mu_0\in R$ such that $Q-\mu B^T B<0$ holds for all $\mu>\mu_0$.

(iii) There exists $F\in R^{n\times m}$ such that $Q+FB+B^T F^T<0$.

The equivalency between (i) and (iii) is frequently used to transform matrix inequalities into LMI.

2. Stability analysis using polynomial Lyapunov function

In this section the robust stability condition is developed for discrete-time polytopic system using polynomially parameter dependent Lyapunov function (PPDLF). The developments pursue the ideas presented in (Ebihara *et al*. 2006) and (Vesely, 2007).

Substituting PPDLF (9) into the robust stability condition (5) we obtain:

$$A(\alpha)^T G(\alpha,p)^T \Pi_p(\alpha) G(\alpha,p) A(\alpha) - G(\alpha,p)^T \Pi_p(\alpha) G(\alpha,p) < 0 \tag{11}$$

Inequality (11) can be rewritten as

$$\begin{bmatrix} G(\alpha,p) \\ G(\alpha,p)A(\alpha) \end{bmatrix}^T \begin{bmatrix} -\Pi_p(\alpha) & 0 \\ 0 & \Pi_p(\alpha) \end{bmatrix} \begin{bmatrix} G(\alpha,p) \\ G(\alpha,p)A(\alpha) \end{bmatrix} < 0 \tag{12}$$

The polynomial term $G(\alpha,p)$ can be eliminated using Lemma 4 and the following equality (Ebihara *et al*. 2006)

$$\begin{bmatrix} G(\alpha,p) \\ G(\alpha,p)A(\alpha) \end{bmatrix} = \begin{bmatrix} A(\alpha)G(0,p)^T & -G(0,p)^T \\ L(M(\alpha),p) & 0 \\ 0 & L(M(\alpha),p) \end{bmatrix}^{\perp} \qquad (13)$$

where

$$L(M(\alpha),p) = [I_p \otimes M(\alpha) \quad 0_{pn\times n}] + [0_{pn\times n} \quad -I_p \otimes I_n]$$

Owing to equivalency of (i) and (iii) in Lemma 4, and (13), robust stability condition (12) is equivalent to the existence of a matrix $Z \in R^{2(p+1)n\times(2p+1)n}$ such that the following inequality holds

$$\begin{bmatrix} -\Pi_p(\alpha) & 0 \\ 0 & \Pi_p(\alpha) \end{bmatrix} + Z \begin{bmatrix} A(\alpha)G(0,p)^T & -G(0,p)^T \\ L(M(\alpha),p) & 0 \\ 0 & L(M(\alpha),p) \end{bmatrix} +$$
$$+ \begin{bmatrix} A(\alpha)G(0,p)^T & -G(0,p)^T \\ L(M(\alpha),p) & 0 \\ 0 & L(M(\alpha),p) \end{bmatrix}^T Z^T < 0 \qquad (14)$$

For given $M(\alpha)$ inequality (14) is LMI for unknown matrices $\Pi_p(\alpha)$ and *Z*, since $\Pi_p(\alpha)$ is affine in α as well as $A(\alpha)$ and $L(M(\alpha),p)$. Therefore inequality (14) is robust stability condition for uncertain discrete-time polytopic system (1), (2) as stated in Lemma 4.

Lemma 4

Uncertain system (1), is robustly stable in uncertainty domain (2) if there exist matrix *Z* such that inequality (14) holds.

To illustrate the use of Lemma 4 to robust stability analysis we study in detail the case when PPDLF of degree 1 is considered.

Robust stability analysis using PPDLF with p=1

In this section the robust stability condition (14) is formulated for PPDLF degree *p*=1. In this case we consider PPDLF in the form (9) with

$$\Pi_1(\alpha) = \sum_{i=1}^{N} \alpha_i P_i = \sum_{i=1}^{N} \begin{bmatrix} P_{i11} & P_{i12} \\ P_{i21} & P_{i22} \end{bmatrix} \alpha_i \qquad (15)$$

$$G(M(\alpha),1) = \begin{bmatrix} I_n \\ M(\alpha) \end{bmatrix}. \qquad (16)$$

$$M(\alpha) = \sum_{i=1}^{N} \alpha_i M_i \qquad (17)$$

Since (14) is LMI for unknown matrices $\Pi_p(\alpha)$ and *Z*, owing to the linearity of (15) and (17), inequality (14) can be rewritten for each vertex of the polytopic domain and result is obtained by solving the respective *N* LMIs .

$$\begin{bmatrix} -P_i & 0 \\ 0 & P_i \end{bmatrix} + Z \begin{bmatrix} A_i G(0,1)^T & -G(0,1)^T \\ L(M_i,1) & 0 \\ 0 & L(M_i,1) \end{bmatrix} +$$
$$+ \begin{bmatrix} A_i G(0,1)^T & -G(0,1)^T \\ L(M_i,1) & 0 \\ 0 & L(M_i,1) \end{bmatrix}^T Z^T < 0$$
$$i = 1,\dots,N \qquad (18)$$

where

$$G(0,1) = \begin{bmatrix} I_n \\ 0_n \end{bmatrix}, \quad L(M_i,1) = [M_i \quad -I_n]$$

$Z = \{Z_{ij}\}$, i = 1,2,3,4; j = 1,2,3; $Z_{ij} \in R^{n\times n}$

The resulting LMI is

$$Q_{1i} + Q_{2i} + Q_{2i}^T < 0, \quad i = 1,\dots,N \qquad (19)$$

where

$$Q_{1i} = \begin{bmatrix} -P_{i11} & -P_{i12} & 0_n & 0_n \\ -P_{i21} & -P_{i22} & 0_n & 0_n \\ 0_n & 0_n & P_{i11} & P_{i12} \\ 0_n & 0_n & P_{i21} & P_{i22} \end{bmatrix}$$

$$Q_{2i} = \begin{bmatrix} Z_{11}A_i + Z_{12}M_i & -Z_{12} & -Z_{11} + Z_{13}M_i & -Z_{13} \\ Z_{21}A_i + Z_{22}M_i & -Z_{22} & -Z_{21} + Z_{23}M_i & -Z_{23} \\ Z_{31}A_i + Z_{32}M_i & -Z_{32} & -Z_{31} + Z_{33}M_i & -Z_{33} \\ Z_{41}A_i + Z_{42}M_i & -Z_{42} & -Z_{41} + Z_{43}M_i & -Z_{43} \end{bmatrix}$$

Robust stability condition (19) is more general than ones received using linear PDLF (Lemma 2 and Lemma 3). The condition (8) based on PDLF is received for *p*=0: having *p*=0 there is $G(0,0) = I_n$, $L(M_i,0) = M_i$ and

$$Z = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \end{bmatrix}.$$

Taking $M_i = 0_n, Z_{11} = H, Z_{21} = G$, and substituting into (14) the inequality (8) is immediately obtained.

Note that even for *p*=1, when taking constant *M* for the whole uncertainty domain, again the PDLF case is received.

In the next section the obtained PPDLF based robust stability condition (19) is checked and compared with PDLF based robust stability conditions.

3. Examples

The robust stability condition developed in Section 3 with polynomially parameter dependent LF - inequality (19) for polynomial of degree 1, is compared with previous PDLF stability conditions (7) and (8). The comparison includes:

- quadratic stability (4) denoted as **QS**

- robust stability with PDLF (linear Lyapunov function):
  - condition (7) – denoted as **PDLF1**
  - condition (8) – denoted as **PDLF2**
- robust stability with polynomial PDLF:
  - condition (19) – denoted as **PPDLF**.

The matrices of uncertain system (1), (2) has been randomly generated for various input data. We focus on cases close to stability bound.

Characteristics of the examples and generated systems:

- Number of generated uncertain systems in the considered testing data (s=100 or 500)
- System dimension *n* =3 or 4
- Number of polytope vertices *N* = 4 or 8
- "Closeness" to stability bound: maximal eigenvalue modulus of all vertices ρ_M : prescribed interval for ρ_M .

Example 1

s = 500, *n* =3, *N* = 4, $\rho_M \in (0.96; 1)$

Example 2

s = 100, *n* =4, *N* = 4, $\rho_M \in (0.8; 1)$

Example 3

s = 100, n =4, N = 4, $\rho_M \in (0.9; 1)$

Example 4

s = 100, n =2, N = 8, $\rho_M \in (0.9; 1)$

The respective results are summarized in Tab.1

	QS	PDLF1	PDLF2	PPDLF
Ex. 1	1	324	441	455
Ex. 2	0	75	93	96
Ex. 3	0	57	88	89
Ex. 4	0	77	94	94

Tab.1 Comparison of stability results for generated examples

Matrices $M(\alpha)$ has been considered as in (17), taking $M_i = iI_n$ or $M_i = A_i$.

It can be noted that quadratic stability does not work in the studied examples – it is overly conservative to analyse stability for the studied examples. Stability condition PDLF1 often provides reasonable results and is computationally very simple. PDLF2 and PPDLF provides similar results; though there exist cases where PPDLF is better, there are quite rare.

Comparing the computational demands of the compared methods, there are significant differences between them. Though only polynomial of 1st degree is considered, PPDLF is much more demanding. Comparison of number of inequalities and variables to be computed from LMI is summarized in Tab.2.

Conclusion

Robust stability analysis problem has been studied for discrete-time uncertain system. Several methods from literature using parameter dependent Lyapunov function of the linear type has been considered and compared with polynomially parameter dependent Lyapunov function stability analysis method. Though the novel PPDLF method is more general than considered PDLF ones, since it includes the latter as a special case, the obtained results favour the PDLF2 robust stability method. The reason is that PDLF2 provides results rather close to that of PPDLF with significantly less computational demand. According to our experience, PDLF2 can be the first choice; PPDLF can be recommended for the cases rather close to stability bound, where PDLF2 does not indicate stability.

The obtained results for robust stability analysis are of interest also for robust controller design.

	number of unknown variables	number of inequalities to be solved
QS	$\frac{n(n+1)}{2}$	$\frac{n(n+1)}{2}N$
PDLF1	$\frac{n(n+1)}{2}N+n^2$	$(2n+1)nN$
PDLF2	$\frac{n(n+1)}{2}N+2n^2$	$(2n+1)nN$
PPDLF – 1st degree	$2(n+1)nN+12n^2$	$(4n+1)2nN$

Tab.2 Comparison of dimensions of problem for various robust stability conditions

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