Algorithms for robust controller design: application to CSTR

Dalibor Puna, Monika Bakošová

Abstract
This paper presents the application of robust static output feedback control (RSOFC) to an exothermic continuous-time stirred tank reactor (CSTR) with parametric uncertainties. Necessary and sufficient conditions for stabilization of a linear continuous-time uncertain polytopic system via static output feedback are given at first. Then the problem of RSOFC design is converted to solution of linear matrix inequalities (LMIs) and two LMI based algorithms, iterative and noniterative ones are used. The design procedure guarantees with sufficient conditions the robust quadratic stability and guaranteed cost. The possibility to use a robust static output feedback for control of CSTRs with uncertainties is demonstrated by simulation results.

Keywords: robust control, static output feedback, LMI, chemical reactor

Introduction

Chemical reactors are ones of the most important plants in chemical industry. Their operation, however, is corrupted with many different uncertainties. Some of them arise from varying or not exactly known parameters, as e.g. chemical kinetics or reaction activity. In other cases, operating points vary. Various types of perturbations also affect chemical reactors. All these uncertainties can cause poor performance or even instability of closed-loop control systems. Application of robust control approach can be one of ways how to overcome all these problems, which may seriously influence control design for chemical reactors, see e.g. Alvarez-Ramirez and Femat (1999), Mikleš et al. (2006a), Bakošová et al. (2006).

Robustness has been recognized as a key issue in the analysis and design of control systems for the last two decades. One of the solved problems is also the problem of a robust static output feedback, which is till now important opened question in control engineering, see e.g. Syrmos and Abdallah (1997) and references therein. Various approaches have been used to study two aspects of he robust stabilization problem. The first aspect is related to conditions under which the linear system described in the state space can be stabilized via output feedback. The necessary and sufficient conditions to stabilize the linear continuous time invariant system via static output feedback can be found e.g. in Kučera and De Souza (1995), Veselý (2004). Recently it has been shown that an extremely wide array of robust controller design problems can be reduced to the problem of finding a feasible point under biaffiane matrix inequality (BMI) constraint. In Veselý (2002), the BMI problem is reduced to LMIs problem. LMIs have been used to design robust output feedback controllers also in Benton and Smith (1999), Yu and Chu (1999) and others. The second aspect of the robust stabilization problem is related to finding a procedure for obtaining a stabilizing or robustly stabilizing control law. Most of the recent works present iterative algorithms in which a set of LMI problems are repeated until certain convergence criteria are met see e.g. Cao and Sun (1998).

In this paper, necessary and sufficient conditions for stabilization of linear continuous time variant systems via static output feedback will be presented at first. The problem of robust controller design with the static output feedback is also reduced to LMI problems, because LMI theory is a powerful tool to design robust output feedback controllers (Boyd (1994)).

Two main approaches to algorithm design, iterative and non-iterative ones, are used in this paper. These approaches are based on linear statespace representation of a controlled system. The design procedure guarantees with sufficient conditions the robust quadratic stability and guaranteed cost. The designed robust PI controllers are used to robust stabilization of a continuous-time stirred tank reactor (CSTR).

1. Problem formulation

Physical model of a controlled system often leads to a state-space description of its dynamical behavior. Resulting state-space equations typically involve physical parameters whose values are only approximately known, as well as approximations of complex and nonlinear phenomena. So an uncertain state-space model is obtained, which can be often expressed in the form of a linear timevarying system.

Consider the LTV system in the form of a polytopic linear differential inclusion (PLDI) (Boyd (1994)).

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{y}(t) &= \mathbf{C}(t)\mathbf{x}(t) \end{aligned} \tag{1}$$

The problem of control with a static output feedback can be formulated as follows. For the system (1) find a proportion of the output signal, which is passed (fed back) to the input used for control of dynamic behavior of the system. In the case of system (1), it is necessary to find a static output feedback such that the closed-loop system (2) is stable.

$$\dot{\mathbf{x}}(t) = \left(\mathbf{A}(t) + \mathbf{B}(t)\mathbf{F}\mathbf{C}(t)\right)\mathbf{x}(t) = \mathbf{A}_{CL}(t)\mathbf{x}(t) \tag{2}$$

Finding of **F** is important when the state matrix **A** is unstable since having **F** leads to a stabilizing static output feedback.

2. Robust output feedback controller design

Consider the uncertain closed-loop system (2) with

$$\mathbf{A}_{CL}(t) \in Co\{\mathbf{A}_{CL1},\dots,\mathbf{A}_{CLn}\} := \left\{\sum_{i=1}^{n}\alpha_i \mathbf{A}_{CLi} : \alpha_i \ge 0, \sum_{i=1}^{n}\alpha_i = 1\right\},$$

$$\mathbf{A}_{CLi} = \mathbf{A}_i + \mathbf{B}_i\mathbf{F}\mathbf{C}_i \qquad (3)$$

A sufficient condition for the asymptotic stability of the system (2) is feasibility, e. a. the existence of a quadratic Ljapunov function $V(\mathbf{x}) = \mathbf{x}(t)^T \mathbf{P}\mathbf{x}(t)$, $\mathbf{P} > 0$ such that its derivation is negative along all state trajectories. If exists a $\mathbf{P} > 0$, system (2) is quadratically stable and following statement holds: system (2) is quadratically stable if and only if there exists a positive definite matrix $\mathbf{P} > 0$ such that following inequalities are satisfied

$$\mathbf{A}_{CLi}^T\mathbf{P} + \mathbf{P}\mathbf{A}_{CLi}^T < 0,\ \mathbf{P} > 0,\ i = 1,\dots,n \qquad (4)$$

Consider the polytopic closed-loop system (2). Then the following three statements are equivalent (Veselý (2002)):

1. The system (2) is simultaneously static output feedback stabilizable with guaranteed cost

$$\int_0^\infty \left(\mathbf{x}(t)^T\mathbf{Q}\mathbf{x}(t) + \mathbf{u}(t)^T\mathbf{R}\mathbf{u}(t)\right)dt \le \mathbf{x}_0(t)^T\mathbf{P}\mathbf{x}_0(t) = J^*$$

$$\mathbf{P} > 0 \qquad (5)$$

2. There exist matrices $\mathbf{P} > 0$, $\mathbf{Q} > 0$, $\mathbf{R} > 0$ and a matrix **F** such that the following inequalities hold

$$\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i - \mathbf{P}\mathbf{B}_i\mathbf{R}^{-1}\mathbf{B}_i^T\mathbf{P} + \mathbf{Q} \le 0,\ i = 1,\dots,n \qquad (6)$$

$$\left(\mathbf{B}_i^T\mathbf{P} + \mathbf{R}\mathbf{F}\mathbf{C}_i\right)\mathbf{\Phi}_i^{-1}\left(\mathbf{B}_i^T\mathbf{P} + \mathbf{R}\mathbf{F}\mathbf{C}_i\right)^T - \mathbf{R} \le 0 \qquad (7)$$

where

$$\mathbf{\Phi}_i = -\left(\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i - \mathbf{P}\mathbf{B}_i\mathbf{R}^{-1}\mathbf{B}_i^T\mathbf{P} + \mathbf{Q}\right),\ i = 1,\dots,n \qquad (8)$$

Further, for the system (2) the following statements are equivalent (Veselý (2002)):

1. The system (2) is static output feedback simultaneously stabilizable.
2. There exist matrices $\mathbf{P} > 0$, $\mathbf{Q} > 0$, $\mathbf{R} > 0$ and a matrix **F** satisfying the following matrix inequalities

$$\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i - \mathbf{P}\mathbf{B}_i\mathbf{R}^{-1}\mathbf{B}_i^T\mathbf{P} - \mathbf{C}_i^T\mathbf{F}^T\mathbf{R}\mathbf{F}\mathbf{C}_i + \mathbf{Q} \le 0,$$
$$i = 1,\dots,n \qquad (9)$$

$$\left(\mathbf{B}_i^T\mathbf{P} + \mathbf{R}\mathbf{F}\mathbf{C}_i\right)\mathbf{\Phi}_i^{-1}\left(\mathbf{B}_i^T\mathbf{P} + \mathbf{R}\mathbf{F}\mathbf{C}_i\right)^T - \mathbf{R} \le 0 \qquad (10)$$

where

$$\mathbf{\Phi}_i = -\left(\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i - \mathbf{P}\mathbf{B}_i\mathbf{R}^{-1}\mathbf{B}_i^T\mathbf{P} - \mathbf{C}_i^T\mathbf{F}^T\mathbf{R}\mathbf{F}\mathbf{C}_i + \mathbf{Q}\right),$$
$$i = 1,\dots,n \qquad (11)$$

The non-iterative design procedure for simultaneous static output feedback stabilization of the system (2) with guaranteed cost is based on statements formulated above and has two basic steps.

1. Using the LMI based algorithm calculates $\mathbf{P} > 0$ from the inequality (6).
2. Using the LMI based algorithm compute **F** from the inequality (7).

The iterative design procedure is based on statements formulated above and has following steps.

1. Using the LMI based algorithm calculates $\mathbf{S}_j = \mathbf{P}_j^{-1} > 0$ from the inequality (9).
2. Using the LMI based algorithm compute the gain matrix $\mathbf{F}_j$ from the inequality (10).
3. Compute $er = \left\|\mathbf{F}_j - \mathbf{F}_{j-1}\right\|$, if $er \le error$ stop else $j = j+1$ and go to step 1

If the solutions of (6), (7) or (9), (10) are not feasible, either the system (2) is not stabilizable with a prescribed guaranteed cost, or it is necessary to change **Q**, **R** and γ in order to find a feasible solutions

3. Simulation results

Consider a continuous-time stirred tank reactor (CSTR) with the first order irreversible parallel exothermic reactions according to the scheme $A \xrightarrow{k_1} B$, $A \xrightarrow{k_2} C$, where *B* is the main product and *C* is the side product. Under the condition of perfect mixing, the dynamic mathematical model of the controlled system has been obtained by mass balances of reactants, energy balance of the reactant mixture and energy balance of the coolant. Using usual simplifications, the model of the CSTR can be described by four nonlinear differential equations

$$\frac{dc_A}{dt} = -\left(\frac{q_r}{V_r} + k_1 + k_2\right)c_A + \frac{q_r}{V_r}c_{Af} \qquad (12)$$

$$\frac{dc_B}{dt} = -\frac{q_r}{V_r}c_B + k_1 c_A + \frac{q_r}{V_r}c_{Bf} \qquad (13)$$

$$\frac{dT_r}{dt} = -\frac{h_1 k_1 + h_2 k_2}{\rho_r c_{pr}}c_A + \frac{q_r}{V_r}(T_{rf} - T_r) + \frac{A_h U}{V_r \rho_r c_{pr}}(T_c - T_r) \qquad (14)$$

$$\frac{dT_c}{dt} = \frac{q_c}{V_c}(T_{cf} - T_c) + \frac{A_h U}{V_c \rho_c c_{pc}}(T_r - T_c) \qquad (15)$$

with initial conditions $c_A(0)$, $c_B(0)$, $T_r(0)$ and $T_c(0)$. Here, *t* is time, *c* are concentrations, *T* are temperatures, *V* are volumes, ρ are densities, c_p are specific heat capacities, *q* are volumetric flow rates, *h* are reaction enthalpies, A_h is the heat transfer area and *U* is the heat transfer coefficient. The subscripts denote $._r$ the reactant mixture, $._c$ the coolant, $._f$ feed values and the superscript $.^s$ the steady-state values.

The reaction rates k_1, k_2 are expressed as

$$k_j = k_{0j}\exp\left(-\frac{E_j}{R T_r}\right),\ j = 1, 2 \qquad (16)$$

where k_0 are pre-exponential factors, *E* are activation energies, *R* is the gas constant. The values of all parameters and feed values are in Table 1.

Variable	Unit	Value
V_r	m^3	0.23
V_c	m^3	0.21
ρ_r	$kg\ m^{-3}$	1020
ρ_c	$kg\ m^{-3}$	998
c_{pr}	$kJ\ kg^{-1}\ K^{-1}$	4.02
c_{pc}	$kJ\ kg^{-1}\ K^{-1}$	4.182
$g_1=E_1/R$	K	9850
$g_2=E_2/R$	K	22019
A_h	m^2	1.51
U	$kJ\ m^{-2}\ min^{-1}\ K^{-1}$	42.8
c_{Af}	$kmol\ m^{-3}$	4.22
c_{Bf}	$kmol\ m^{-3}$	0
T_{rf}	K	310
T_{cf}	K	288
q_r^s	$m^3\ min^{-1}$	0.015
q_c^s	$m^3\ min^{-1}$	0.004

Tab.1 Reactor Parameters and Inputs

Model uncertainties of the over described reactor follows from several facts. At first, there are two only approximately known physical parameters in this reactor, which can change their values around the nominal values as it is shown in Table 2. So we have parametric uncertainties in our system. There are also dynamic uncertainties in the controlled system as according to the values of uncertain parameters vary operating conditions and linearized models of the reactor with different values of matrix coefficients in the state space description are derived. The dynamic uncertainties also include the gap between linearized models and the original nonlinear model or the true physical system.

Variable	Minimal value	Maximal value
k_{10} [min^{-1}]	1.5×10^{11}	1.6×10^{11}
k_{20} [min^{-1}]	4.95×10^{26}	12.15×10^{26}

Tab.2 Uncertain parameters of the reactor

The steady state behavior of the chemical reactor with nominal values of uncertain parameters was studied at first and it is shown in Fig.1. Q_{GEN} is the heat generated by chemical reactions and Q_{OUT} is the heat removed by the jacket and the product stream and they are calculated as follows

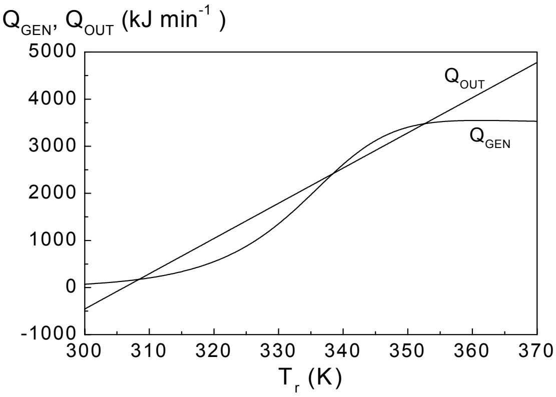
$$Q_{GEN} = -\left(h_1 k_1 c_A + h_1 k_2 c_A\right)V \qquad (17)$$

$$Q_{OUT} = A_h U\left(T_r - T_c\right) + \left(q \rho c_p\right)_r \left(T_r - T_f\right) \qquad (18)$$

From Fig. 1 it is clear, that the reactor has three steady states, two of them are stable and one is unstable. But the maximal concentration of c_B is obtained in the unstable steady state. So the main operating point with maximal production of B is characterized by following values of state variables

$$\left[c_A^s, c_B^s, T_r^s, T_c^s\right] = [1.8614,\ 1.0113,\ 338.41,\ \ 328.06]$$

The steady state behavior of the chemical reactor is similar for all four combinations of minimal and maximal values of 2 uncertain parameters. The maximal concentration of B is always obtained in an unstable steady state. It follows from the steady state analysis of the reactor that it is necessary to stabilize it robustly in the main operating point.



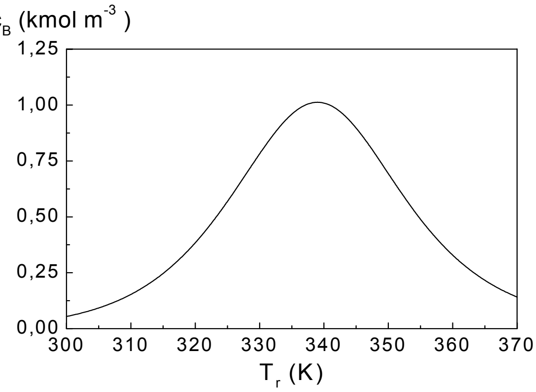


Fig.1 Steady state behavior of the chemical reactor for nominal values of uncertain parameters

Design of a robust stabilizing controller is based on having a linear state space model (1) of the controlled system. Linearized mathematical model has been derived under the assumption that the control inputs are the reactant flow rate q_r and the coolant flow rate q_c and the controlled outputs are the temperature of reaction mixture T_r and temperature of coolant T_c. The other input variables are considered to be constant.

But the output feedback does not have integral action. One way of forcing an integral action to the output feedback is to put a set of integrators at the output of the plant (Mikleš et al. (2006b)).

By introducing

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} c_A - c_A^s \\ c_B - c_B^s \\ T_r - T_r^s \\ T_c - T_c^s \end{pmatrix}$$

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} q - q^s \\ q_c - q_c^s \end{pmatrix}, \ \mathbf{y} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \qquad (19)$$

and with respect on integral part of the plant in the main operating point, the linearized model of the CSTR was obtained in the form (1), where the matrices of the nominal model have the form presented in Puna and Bakošová (2007). For 2 uncertain parameters, we have obtained 2^2 = 4 linearized mathematical models, which differ in coefficients of A_i, B_i. These systems represent vertices of the uncertain polytopic system and they all are unstable. It was further important to find a robust static output feedback, which would be able to stabilize the whole uncertain system with the guaranteed cost expressed by (5). The parameters of matrices Q, R have been chosen according to the values of state variables and control inputs. For finding a stabilizing output feedback PI controller it is necessary to solve sets of LMIs. The non-iterative algorithm evaluates sets of inequalities described by (6) and (7). Competitive iterative algorithm solves inequalities described by (9), (10). Each set consists of 4 LMIs. The feasibility of the solutions assures that the reactor is robust static output feedback quadratically stabilizable and PI controllers robustly stabilize with guaranteed cost the whole uncertain system. For solving the LMIs, the YALMIP toolbox with SeDuMi solver has been used (Löfberg (2004)). There are parameters, which influence solution and can be changed: Q, R and γ . In dependence on the choice of these parameters, it was possible to find several stabilizing PI controllers, which stabilize the polytopic system with 4 vertices and also stabilize the reactor. Two of these PI controllers were chosen for simulation tests and they are presented in the Table 3.

	non-iterative	iterative
q_{const}	1	1
r_{const}	1	0.01
γ	0.1	0.01
F^T	$\begin{pmatrix} 0.076 & -0.0040 \\ -0.0089 & 0.0849 \\ 0.0356 & -0.0003 \\ -0.0029 & 0.0366 \end{pmatrix}$	$\begin{pmatrix} 0.078 & -0.0030 \\ 0.0015 & 0.0442 \\ 0.0047 & 0.0022 \\ 0.0016 & 0.0468 \end{pmatrix}$
J^*	11.4378	5.7166

Tab.3 Parameters of stabilizing PI controllers

For all stabilizing PI controllers all closed loop systems obtained for the nominal system and also for all vertices of the polytopic system are stable, e. a. all eigenvalues of all state matrices of all closed loop systems have negative real parts. But for all systems some of eigenvalues are complex and so we have periodic output by control.

Further, it was supposed that a following disturbance in the feed temperature of the reaction mixture occurs: T_{rf} increases by 4K for $t \in [30\ \text{min}, 40\text{min}]$.

Simulation results obtained with the robust static feedback PI controller designed using the non-iterative algorithm is shown in Fig. 2. Here, the reactor represented by its nonlinear model with the disturbance described above was controlled. Fig. 3 represents simulation results obtained with PI controller designed by the iterative algorithm. Here, also the reactor represented by its nonlinear model with the disturbance described above was controlled.

It is clear from these simulation experiments that the robust static output feedback PI controllers, designed by both methods, are able to stabilize the CSTR with uncertainties towards the main operating point from any initial conditions. The controllers are able to manage also a small and short disturbance. The control responses of the CSTR with used PI controllers are compared according to the values of the cost function *J*. Its value computed for the control response with PI controller obtained by the iterative algorithm is less than the values computed for the control response with PI controller obtained by the non-iterative algorithm. On the other hand, using of PI controller obtained by the iterative algorithm leads to slower control response and higher sensibility to disturbances.

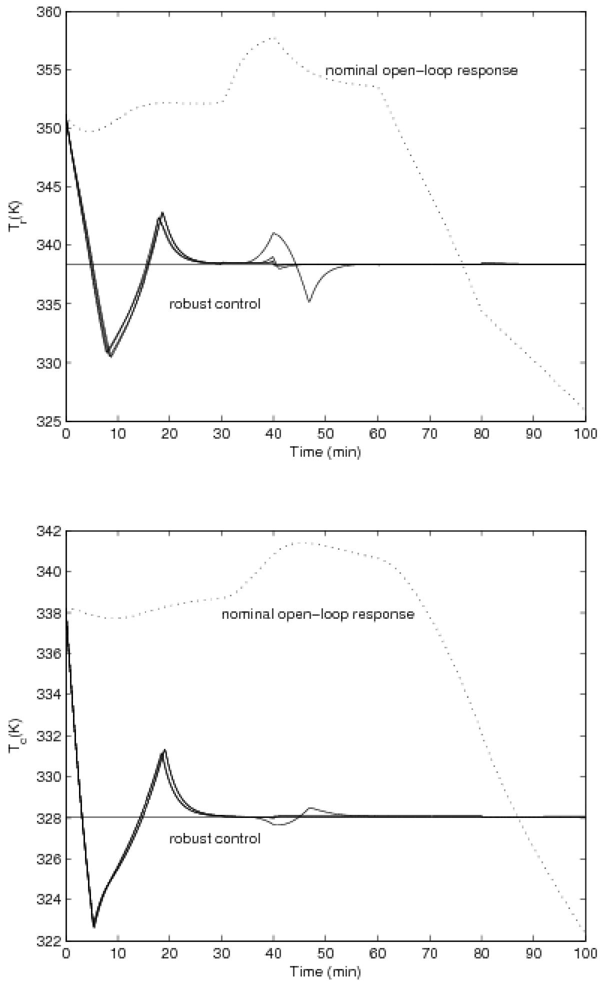


Fig.2 RSOFC of the CSTR – temperature of the reaction mixture and temperature of coolant in the jacket of the reactor (non – iterative algorithm)

4. Conclusion

In this paper, the possibility to stabilize an exothermic chemical reactor with uncertainties working in the unstable operating point via static output feedback controllers is studied. The robust controllers design is converted to solving of LMI problems. A computationally LMI based non-iterative and iterative algorithms were used for the design of robust static output feedback PI controllers. These algorithms are based on linear state-space representation of a controlled system. The design procedures guarantee with sufficient conditions the robust quadratic stability and guaranteed cost. The designed robust controllers are able to stabilize the exothermic CSTR for the entire operating area not only for a single operating point.

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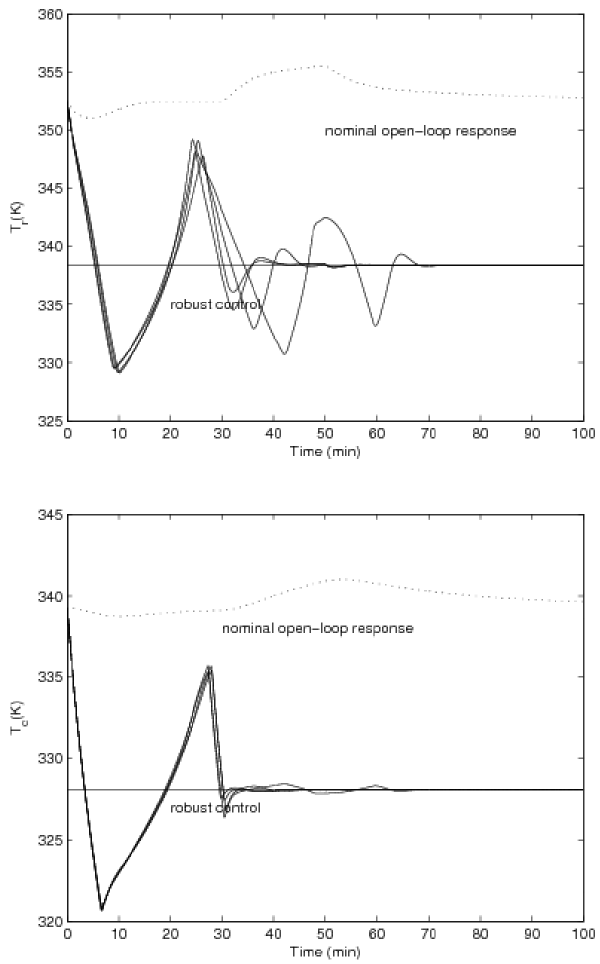

Fig.3 RSOFC of the CSTR - temperature of the reaction mixture and temperature of coolant in the jacket of the reactor (iterative algorithm)

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Ing. Dalibor Puna
Doc. Ing. Monika Bakošová, CSc.

Slovak University of Technology in Bratislava
Faculty of Chemical and Food Technology
Institute of Information Engineering, Automation and Mathematics
Department of Information Engineering and Process Control
Radlinského 9
812 37 Bratislava
Tel: +421 2 59325364
Fax: +421 2 52496469

dalibor.puna@stuba.sk, monika.bakosova@stuba.sk