Control of systems with parametric uncertainties using a robust PI controller

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Abstract
A method for design of robust PI controllers is presented in this paper. The method is based on plotting the stability boundary locus in (k_p,k_i) – plane and then parameters of stabilizing PI controllers are obtained. Designed PI controllers stabilize controlled systems with interval parametric uncertainties. The Kharitonov theorem is used for stability analysis of the designed feedback control loop. Robustness of a designed PI controller is verified by simulations.

Key words: Robust control, parametric uncertainty, PI controller

Introduction

There has been done a great amount of research work on the tuning of PID controllers since these types of controllers have been widely used in industrial applications, see e.g. [1]. PID controller design in classical control engineering is based on a plant with fixed parameters, see e.g. [5].

In the real world, however, most process models are not known exactly and so, models contain uncertainties. Hence control system design for both stability and performance robustness always requires taking uncertainties into account. This requirement has attracted the attention of many researches over the years to find solutions for the problems of robust stability analysis and robust controller synthesis for systems with uncertainties, see e.g. [3], [2]. One of the solved problems is also the problem of robust controller synthesis for systems with parametric uncertainty.

In this paper, a method for design of robust PI controllers is presented [6]. The method is based on plotting the stability boundary locus in the (k_p,k_i) – plane and then parameters of a stabilizing PI controller are calculated. The PI controller stabilizes a controlled system with interval parametric uncertainties.

The approach is used for design of a robust PI controller, which is then implemented for control of a laboratory process [7]. The laboratory process is modelled as the 2nd order system with interval uncertainties and this paper presents simulation results. The Kharitonov theorem is used for stability analysis of the designed feedback control loop [4].

1. Notation of an uncertain system

Consider a system with real parametric uncertainty described by the transfer function

$$G(s,q)=\frac{b(s,q)}{a(s,q)} \qquad (1)$$

where q is a vector of uncertain parameters and b, a are polynomials in s with coefficients which depend on q.

An uncertain polynomial

$$a(s,q)=\sum_{i=0}^{n} a_i(q)s^i \qquad (2)$$

is said to have an independent uncertainty structure if each component q_i of q enters into only one coefficient.

A family of polynomials

$$A=a(\cdot,q): q\in Q \qquad (3)$$

is said to be an interval polynomial family if $a(s,q)$ has an independent uncertainty structure, each coefficient depends continuously on q and Q is a box. An interval polynomial family A arises from the uncertain polynomials described by $a(s,q)$ with uncertainty bounds $|q|\le 1$ for $t=0,\dots,n$. When dealing with an interval family, the shorthand notation

$$a(s,q)=\sum_{i=0}^{n}\left[a_i^-,a_i^+\right]s^i \qquad (4)$$

can be used with $[\,a_i^-,a_i^+\,]$ denoting the bounding interval for the i – th component of uncertainty a_i.

2. PI controller design

Consider the single-input single-output (SISO) control system shown in Fig. 1, where

$$G(s)=\frac{N(s)}{D(s)} \qquad (5)$$

is the plant to be controlled and $C(s)$ is a PI controller in the form

$$C(s)=k_p+\frac{k_i}{s}=\frac{k_p s+k_i}{s} \qquad (6)$$

The problem is to find the parameters of the PI controller of (6), which stabilize the system in Fig. 1, where w is the set point, e – control error, u – control input and y – controlled output.

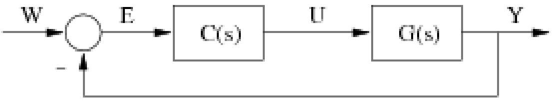

Fig.1 Control system

Decomposing the numerator and the denominator polynomials of (5) into their even and odd parts, and substituting $s = j\omega$, gives

$$G(j\omega) = \frac{N_e(-\omega^2) + j\omega N_o(-\omega^2)}{D_e(-\omega^2) + j\omega D_o(-\omega^2)} \tag{7}$$

The closed loop characteristic polynomial can be written as

$$\begin{aligned}\Delta(j\omega) = &\left[k_i \mathrm{N_e}(-\omega^2) - k_p\omega^2 N_o(-\omega^2) - \omega^2 D_o(-\omega^2)\right] + \\ &+ j\left[k_p\omega N_e(-\omega^2) + k_i\omega N_o(-\omega^2) + \omega D_e(-\omega^2)\right]\end{aligned} \tag{8}$$

Then, equating the real and imaginary parts of $\Delta(j\omega)$ to zero, one obtains

$$k_p\left(-\omega^2 N_o(-\omega^2)\right) + k_i\left(N_e(-\omega^2)\right) = \omega^2 D_o(-\omega^2) \tag{9}$$

and

$$k_p\left(N_e(-\omega^2)\right) + k_i\left(N_o(-\omega^2)\right) = -D_e(-\omega^2) \tag{10}$$

Let

$$\begin{aligned}F(\omega) &= -\omega^2 N_o(-\omega^2)\\ G(\omega) &= N_e(-\omega^2)\\ H(\omega) &= N_e(-\omega^2)\\ I(\omega) &= N_o(-\omega^2)\\ J(\omega) &= \omega^2 D_o(-\omega^2)\\ K(\omega) &= -D_e(-\omega^2)\end{aligned} \tag{11}$$

Then, (9) and (10) can be written as

$$\begin{aligned}k_p F(\omega) + k_i G(\omega) &= J(\omega)\\ k_p H(\omega) + k_i I(\omega) &= K(\omega)\end{aligned} \tag{12}$$

From these equations parameters of the PI controller (6) are

$$k_p = \frac{J(\omega)I(\omega) - K(\omega)G(\omega)}{F(\omega)I(\omega) - G(\omega)H(\omega)} \tag{13}$$

and

$$k_i = \frac{K(\omega)F(\omega) - J(\omega)H(\omega)}{F(\omega)I(\omega) - G(\omega)H(\omega)} \tag{14}$$

Solving these two equations simultaneously, the stability boundary locus, $l(k_p, k_i, \omega)$, in (k_p,k_i) – plane can be obtained. The stability boundary locus divides the parameter plane ((k_p,k_i) – plane) into stable and unstable regions. Choosing a test point within each region, the stable region which contains the values of stabilizing k_p and k_i parameters can be determined [6].

The method is very fast and effective, however, frequency rating becomes important. An efficient approach to avoid frequency rating can be obtained by using the Nyquist plot. It is only necessary to find real values of ω that satisfy

$$\mathrm{Im}[G(s)] = 0 \tag{15}$$

where $s = j\omega$.

3. Kharitonov's Theorem

In order to use the Kharitonov's theorem for robust stability, polynomials associated with an interval polynomial family A have to be defined at first. In the definition below the polynomials are fixed in the sense that only the bounds a_i^- and a_i^+ enter into the description but not the a_i themselves. The number of polynomials is four and they are independent on the degree of $a(s,q)$. Associated with the interval polynomial family (4) are four fixed Kharitonov polynomials [4]

$$\begin{aligned}K_1 &= a_0^- + a_1^- s + a_2^+ s^2 + a_3^+ s^3 + \dots\\ K_2 &= a_0^+ + a_1^+ s + a_2^- s^2 + a_3^- s^3 + \dots\\ K_3 &= a_0^+ + a_1^- s + a_2^- s^2 + a_3^+ s^3 + \dots\\ K_4 &= a_0^- + a_1^+ s + a_2^+ s^2 + a_3^- s^3 + \dots\end{aligned} \tag{16}$$

The interval polynomial family A with invariant degree is robustly stable if and only if its four Kharitonov polynomials (16) are stable [4].

4. Stabilization of a plant with interval parametric uncertainty

The stability boundary locus is used to find all the values of the parameters of a PI controller for which the given plant with interval parametric uncertainty is Hurwitz stable [6].

Consider a feedback system with a PI controller (6) and an interval plant

$$G(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \dots + b_0}{a_m s^m + a_{m-1}s^{m-1} + \dots + a_0} \tag{17}$$

where $b_i \in \left[b_i^-, b_i^+\right]$, $i = 0,1,2,\dots,m$ and $a_i \in \left[a_i^-, a_i^+\right]$, $j = 0,1,2,\dots,n$. Let the Kharitonov polynomials associated with N(s) and D(s) is respectively:

$$\begin{aligned}N_1 &= b_0^- + b_1^- s + b_2^+ s^2 + b_3^+ s^3 + \dots\\ N_2 &= b_0^+ + b_1^+ s + b_2^- s^2 + b_3^- s^3 + \dots\\ N_3 &= b_0^+ + b_1^- s + b_2^- s^2 + b_3^+ s^3 + \dots\\ N_4 &= b_0^- + b_1^+ s + b_2^+ s^2 + b_3^- s^3 + \dots\end{aligned} \tag{18}$$

and

$$\begin{aligned}D_1 &= a_0^- + a_1^- s + a_2^+ s^2 + a_3^+ s^3 + \dots\\ D_2 &= a_0^+ + a_1^+ s + a_2^- s^2 + a_3^- s^3 + \dots\\ D_3 &= a_0^+ + a_1^- s + a_2^- s^2 + a_3^+ s^3 + \dots\\ D_4 &= a_0^- + a_1^+ s + a_2^+ s^2 + a_3^- s^3 + \dots\end{aligned} \tag{19}$$

By taking all combinations of the $N_i(s)$ and $D_j(s)$ for $i, j = 1,2,3,4$, the following family of sixteen Kharitonov plants can be obtained

$$G_K(s) = G_{ij}(s) = \frac{N_i(s)}{D_j(s)} \tag{20}$$

where $i, j = 1,2,3,4$.

Define the set $S(C(s)G(s))$, which contains all values of the parameters of the controller $\mathrm{C(s)}$ which stabilizes $\mathrm{G(s)}$. Then the set of all the stabilizing values of parameters of a PI controller, which stabilizes the interval plant of (17) can be written as

$$\begin{aligned} S(C(s)G_K(s)) &= S(C(s)G_{11}(s))\cap \\ &\cap S(C(s)G_{12}(s))\cap\cdots\cap S(C(s)G_{44}(s)) \end{aligned} \tag{21}$$

4.1 Robust PI controller design for a laboratory process

The laboratory process was identified in the form of a transfer function [7]

$$G(s)=\frac{-b_1 s+b_0}{a_2 s^2+a_1 s+a_0} \tag{22}$$

where $b_1 \in [0.3589, 11.3142]$, $b_0 \in [0.3700, 5.1900]$, $a_2 \in [86.3200, 122.6200]$, $a_1 \in [18.5800, 22.1500]$ and $a_0 = 1$. The objective is to calculate all parameters of PI controllers, which stabilize $\mathrm{G(s)}$ (22). Consider the first Kharitonov plant ($i=1$ and $j=1$), which is

$$G_{11}(s)=\frac{-0.3589s+0.37}{86.32s^2+18.58s+1} \tag{23}$$

The closed loop characteristic polynomial has according to (8) the form

$$\Delta j\omega=\left[-a_1\omega^2+b_1k_p\omega^2+b_0k_i\right]+j\left[-a_2\omega^3+\left(b_0k_p-b_1k_i+1\right)\omega\right] \tag{24}$$

and then

$$\begin{aligned} k_p &= \frac{\omega^2(a_2b_0+b_1a_1)-b_0}{b_0^2+b_1^2\omega^2}=\frac{38.6068\omega^2-0.37}{0.1288\omega^2+0.1369} \\ k_i &= \frac{(a_1-b_1k_p)\omega^2}{b_0}=\frac{(18.58-0{,}3589k_p)\omega^2}{0.37} \end{aligned} \tag{25}$$

Since $\mathrm{Im}[G_{11}(j\omega)=0]$ is satisfied for $\omega=0.48321 \quad rad/s$, it is necessary to obtain stability boundary locus for $\omega \in (0, 0.48321)$. All stabilizing values of k_p and k_i are shown in Fig. 2.

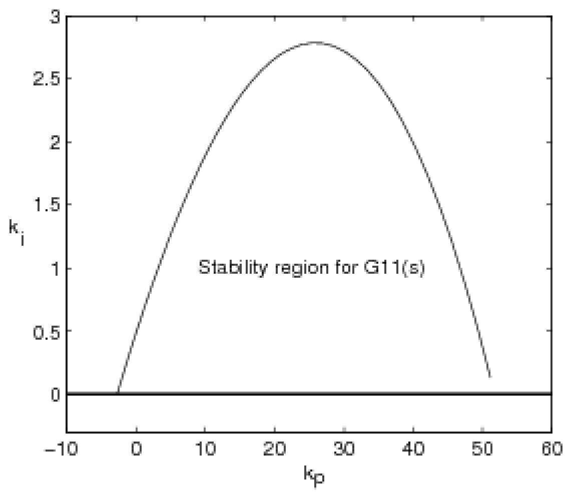

Fig.2 Stability region for $G_{11}(s)$

The stability regions of sixteen Kharitonov plants are shown in Fig. 3, where intersection of these regions represents the stability region. The stability region is in more detail shown in Fig. 4.

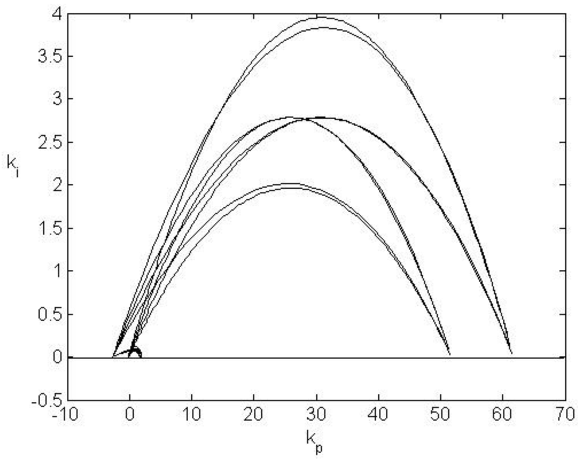

Fig.3 Stability regions for sixteen Kharitonov plants

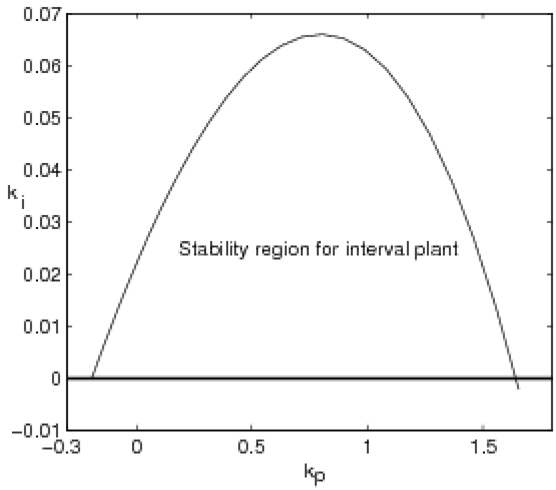

Fig.4 Intersection of stability regions for sixteen Kharitonov plants

4.1 Simulation results

The parameters k_p and k_i of the PI controller were chosen according to the Fig. 4 as it is expressed in (26)

$$C(s)=\frac{k_p s+k_i}{s}=\frac{0.5s+0.02}{s} \tag{26}$$

The robust stability of the closed loop with PI controller (26) was tested using Kharitonov's Theorem. The result of this test is, that the closed loop with controller $\mathrm{C(s)}$ in the form (26) and the controlled system is robustly stable. Robustness of the designed PI controller (26) was verified also by simulations. The controlled process was represented by the nominal model and models obtained for minimal and maximal values of parameters. Simulation results obtained for nominal model and two models with minimal and maximal values of parameters are shown in Fig. 5.

Conclusion

The method based on plotting the stability boundary locus in the (k_p,k_i) – plane was used for robust PI controller design for a system with interval uncertainty. Presented approach is simple and does not need e.g. using optimization methods to solve a set of inequalities.

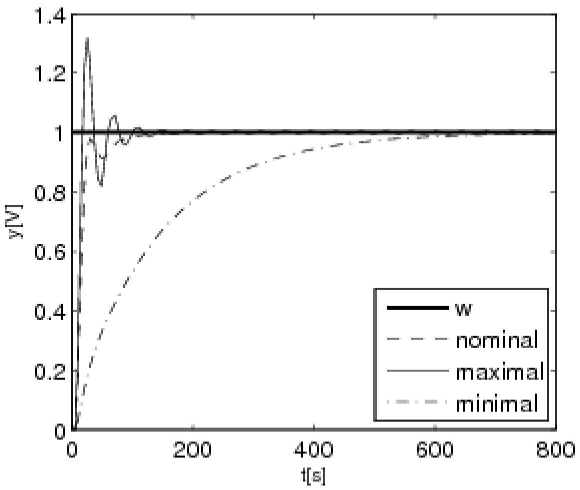

Fig.5 Control responses of a system with interval parametric uncertainty

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References

[1] ÅSTRÖM, K.J, HÄGLUND, T.: PID Controllers: Theory, Design and Tuning. Instrument Society of America, Research Triangle Park, NC, second edition, 1995.

[2] BAKOŠOVÁ, M., PUNA, D., MÉSZÁROS: Control of a continuous-time stirred tank reactor via robust static output feedback. In VPrep. 14th Meditteranean Conference on Control and Automation, Acona, Italy, 2006.

[3] BARMISH, B.: New Tools for Robustness of Linear Systems. Macmillan Publishing Company, New York, 1994.

[4] KHARITONOV, V.L.: Asymptotic stability of an equilibrium position of a family systems of linear differential equations. Differential Equations, 14: 2086-2088, 1978.

[5] MIKLEŠ, J., FIKAR, M: Process modelling, identification and control II. STU Press, Bratislava, 2004.

[6] TAN, N., KAYA, I.: Computation of stabilizing PI controllers for interval systems. In Meditteranean Conference on Control and Automation, Rhodes, Greece, June 2003.

[7] VANEKOVÁ, K, BAKOŠOVÁ, M., ZÁVACKÁ, J.: Robust control of a laboratory process using control system SIMATIC. In Proc. 34th Int. Conf. SSCHE, Tatranské Matliare, Slovakia, 2007.

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